Theorem 4.4.2: For any number field K of degree n over $\mathbb{Q}$ we have

$$\boxed{\sqrt{|d_K|} \geqslant \frac{n^n}{n!} \cdot \left(\frac{\pi}{4}\right)^{n/2}.}$$

$K_{\mathbb{R}} = \{ (z_\sigma) \in \mathbb{C}^{\Sigma} \mid \forall \sigma : z_{\bar\sigma} = \bar z_\sigma \}$

Proof: For $t > 0$ set $X_t := \{ (z_\sigma)_{\sigma \in \Sigma} \in K_{\mathbb{R}} \mid \sum_\sigma |z_\sigma| < t \}$. convex centrally symmetric

Claim 1: $\boxed{\mathrm{vol}(X_t) = 2^r \cdot \pi^s \cdot \frac{t^n}{n!}}$

Proof: $X_t \mapsto \{ (x_1, \dots, x_r, z_1, \dots, z_s) \in \mathbb{R}^r \times \mathbb{C}^s \mid \sum_{i=1}^r |x_i| + 2 \sum_{i=1}^s |z_i| < t \} =: Y_t$

$\Rightarrow \boxed{\mathrm{vol}(X_t) = 2^s \cdot \mathrm{vol}(Y_t)}$ wrt the usual measure.

$dx\,dy = \rho \cdot d\rho\, d\theta$

$z = x + iy = \rho e^{i\theta}$

$\boxed{\mathrm{vol}(Y_t)} = \int \cdots \int_{\substack{\sum |x_i| + 2\sum |\rho_i| < t \\ \rho_i \geq 0 \\ 0 \leq \theta_i \leq 2\pi}} 1 \cdot \prod_{i=1}^r dx_i \cdot \prod_{i=1}^s (\rho_i \, d\rho_i \, d\theta_i) = (2\pi)^s \cdot 2^r \cdot \int_{\substack{\text{all } x_i, \rho_i \geq 0 \\ \sum x_i + 2\sum \rho_i < t}} \rho_1 \cdots \rho_s \cdot dx_1 \cdots dx_r \cdot d\rho_1 \cdots d\rho_s.$

$2\rho_j = y_j$

$= \boxed{(2\pi)^s \cdot 2^r \cdot 4^{-s} \cdot} \int_{\substack{\text{all } x_i, y_j \geq 0 \\ \sum x_i + \sum y_j < t}} y_1 \cdots y_s \, dx_1 \cdots dx_r \, dy_1 \cdots dy_s.$

$\shortparallel$ $\boxed{I_{r,s}(t)}$

Then $I_{r,s}(t) = t^{r+2s} \cdot I_{r,s}(1) = t^n \cdot I_{r,s}(1)$

$$r > 0 \Rightarrow I_{r,s}(1) = \int_{0 \le x_1 \le 1} \Big(\int_{\sum_{i=2}^{r} x_i + \sum z_j < 1 - x_1} \cdots \Big) dx_1$$

$$= \int_0^1 I_{r-1,s}(1-x_1)\, dx_1 = \int_0^1 (1-x_1)^{n-1} \cdot I_{r-1,s}(1) \cdot dx$$

$$= -\frac{(1-x)^n}{n} \cdot I_{r-1,s} \Big|_0^1 = \frac{1}{n} \cdot I_{r-1,s} .$$

$$\Rightarrow \quad I_{r,s}(1) = \frac{1}{n(n-1)\cdots(2s+1)} \cdot I_{0,s}(1) = \frac{2s!}{n!} \cdot I_{0,s}(1)$$

$$s > 0 \Rightarrow I_{0,s}(1) = \frac{1}{2s} \cdot \frac{1}{2s-1} \cdot I_{0,s-1}(1)$$

$$\Rightarrow I_{0,s}(1) = \frac{1}{(2s)!} \cdot I_{0,0}(1) = \frac{1}{(2s)!}$$

$$\boxed{I_{r,s}(1) = \frac{1}{n!}}$$

$$\Rightarrow m(X_t) = 2^r \cdot (2\pi)^s \cdot 2^n \cdot 4^{-s} \cdot I_{r,s}(t)$$

$$= \pi^s \cdot 2^r \cdot t^{r+s} \cdot I_{r,s}(1)$$

$$= \frac{\pi^s \cdot 2^r}{n!} \cdot t^{n+s}$$

qed (Claim 1).

For $\varepsilon > 0$ choose $t > 0$ such that $\underline{t^n = n! \cdot \left(\frac{4}{\pi}\right)^s \cdot \sqrt{|d_K|} + \varepsilon}$.

$$\Rightarrow \operatorname{vol}(X_t) = 2^n \cdot \pi^s \left[\left(\frac{4}{\pi}\right)^s \cdot \sqrt{|d_K|} + \varepsilon \right] > 2^n \cdot \sqrt{|d_K|} = 2^n \cdot \operatorname{vol}\left(K_{\mathbb{R}} / j(\mathcal{O}_K)\right)$$

$$\Rightarrow \exists\, a \in \mathcal{O}_K \setminus \{0\} : j(a) \in X_t .$$

Then $1 \leq |N_{K/\mathbb{Q}}(a)| = \prod_{\sigma \in \Sigma} |\sigma(a)|$.

because $j(a) \in X_t$.

$$\Rightarrow 1 \leq |N_{K/\mathbb{Q}}(a)|^{1/n} = \left(\prod_{i=1}^{n} |\sigma_i(a)|\right)^{1/n} \leq \frac{1}{n} \cdot \sum_{i=1}^{n} |\sigma_i(a)| \leq \frac{t}{n} .$$

$$\Rightarrow \underline{n^n} \leq t^n = \underline{n! \left(\frac{4}{\pi}\right)^s \cdot \sqrt{|d_K|} + \varepsilon}$$

Let $\varepsilon \to 0 \Rightarrow n^n \leq n! \left(\frac{4}{\pi}\right)^s \cdot \sqrt{|d_K|}$

$$\Rightarrow \boxed{\sqrt{|d_K|} \geq \frac{n^n}{n!} \left(\frac{\pi}{4}\right)^s}$$

qed.

<u>Example</u>: $n = 2 \Rightarrow |d_K| \geq \begin{cases} \left(\frac{2^2}{2!}\right)^2 = 4 & \text{if real quadratic} \quad 8 \\ 4 \cdot \left(\frac{\pi}{4}\right)^2 = \pi^2/4 & \text{if imag. quadratic} \quad : \; 4 \end{cases}$

$K = \mathbb{Q}(\sqrt{d})$

$d_K = \begin{cases} d & d \equiv 1 \ (4) \\ 4d & \text{else} \end{cases}$

$a_n :=$ **Lemma:** The numbers $\boxed{\frac{n^n}{n!}\cdot\left(\frac{\pi}{4}\right)^{n/2}}$ for $n \geqslant 2$ are > 1 and tend to ∞ for $n \to \infty$.

Proof: $a_2 = \frac{2^2}{2!}\left(\frac{\pi}{4}\right) = \frac{\pi}{2} > 1$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)!}\cdot\left(\frac{\pi}{4}\right)^{1/2} = \left(\frac{n+1}{n}\right)^n\cdot\sqrt{\frac{\pi}{4}} = \left(1+\frac{1}{n}\right)^n\cdot\sqrt{\frac{\pi}{4}} \geq (1+1+\ldots)\cdot\sqrt{\frac{\pi}{4}} > \underbrace{2\sqrt{\frac{\pi}{4}}}_{\sqrt{\pi}} > 1.$$

$\Rightarrow (a_n)_n$ strictly monotone increasing.

$\Rightarrow \quad a_n \geq a_2\cdot\sqrt{\frac{\pi}{4}}^{\,n-2} \to \infty$ for $n \to \infty$. qed.

Theorem 4.4.3: *(Hermite)* For any c there exist at most finitely many number fields $K/\mathbb{Q}$ with $\boxed{|d_K| \leqslant c}$.

Prf: $|d_K| \leq c \Rightarrow a_n \leq \sqrt{|d_K|} \Rightarrow \boxed{n \leq c'} =$ constant.

$n := [K/\mathbb{Q}]$ Finish with Thm. 4.4.1. qed.

Theorem 4.4.4: *(Minkowski)* For any number field $K \neq \mathbb{Q}$ we have $|d_K| > 1$.

Prf: $4.4.2 \Rightarrow \sqrt{|d_K|} \geq a_2 > 1$ qed.