Reminder:

Lemma 5.1.1: We have a short exact sequence

$$\begin{tikzcd}
 & & (z_\sigma)_\sigma \arrow[rr, mapsto] & & (\log|z_\sigma|)_\sigma & \\
1 \arrow[r] & (S^1)^\Sigma \arrow[r] & (\mathbb{C}^\times)^\Sigma \arrow[rr, "\ell"] & & \mathbb{R}^\Sigma \arrow[r] & 0 \\
 & & \mathcal{O}_K^\times \arrow[rr, two heads] & & \Gamma &
\end{tikzcd}$$

Lemma 5.2.1: The group Γ is a lattice in $\mathbb{R}^\Sigma$.

Consider the homomorphisms

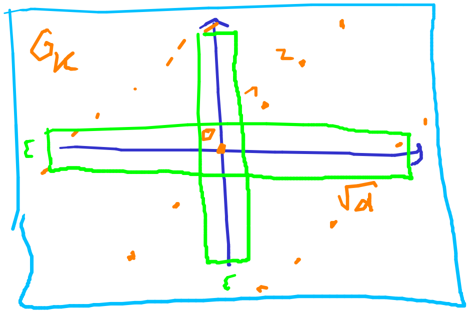
$$\begin{aligned}
\mathrm{Nm}\colon \quad & K_\mathbb{C}^\times = (\mathbb{C}^\times)^\Sigma \longrightarrow \mathbb{C}^\times, \quad (z_\sigma)_\sigma \longmapsto \textstyle\prod_{\sigma\in\Sigma} z_\sigma, \\
\mathrm{Tr}\colon \quad & \cancel{(}\mathbb{R}^{\cancel{\times}}\cancel{)}^\Sigma \longrightarrow \mathbb{R}, \quad (t_\sigma)_\sigma \longmapsto \textstyle\sum_{\sigma\in\Sigma} t_\sigma,
\end{aligned}$$

and the $\mathbb{R}$-subspaces

$$\begin{aligned}
(\mathbb{R}^\Sigma)^+ \quad &:= \quad \big\{(t_\sigma)_\sigma \in \mathbb{R}^\Sigma \;\big|\; \forall\sigma\colon t_{\bar\sigma} = t_\sigma\big\}, \\
H \quad &:= \quad \ker\big(\mathrm{Tr}\colon (\mathbb{R}^\Sigma)^+ \to \mathbb{R}\big).
\end{aligned}$$

Lemma 5.2.3: We have $\Gamma \subset H$ and $\dim_\mathbb{R}(H) = r + s - 1$.

Strategy of proof for $K=\mathbb{Q}(\sqrt{d})$, $d>1$ squarefree. $K \subset \mathbb{R}^2$



Minkowski: $\forall \varepsilon > 0 \quad \exists x \in \mathcal{O}_K \setminus \{0\}$

$0 < |\sigma_1(x)| < \varepsilon$.

and $|\sigma_2(x)| < \dfrac{C}{\varepsilon}$. $\leftarrow$ constant.

$\Rightarrow |N_{L/K}(x)| < C$.

Vary $\varepsilon \Rightarrow$ get ∞ many $x \in \mathcal{O}_K \setminus \{0\}$ with $|N_{L/K}(x)| < C$.

$$|N_{L/K}(x)| = [\mathcal{O}_K : (x)]$$

$\Rightarrow \exists\, \infty$ many distinct $x \in \mathcal{O}_K \setminus \{0\}$ with (x) all equal.

$\Rightarrow$ their ratios are ∞ many units.

5.3 Dirichlet's unit theorem

Theorem 5.3.1: The group Γ is a complete lattice in H.

Proof: To show $\exists\, \Pi \subset H$ bounded with $H = \Gamma + \Pi$.

Take $S := \{ z \in K_\mathbb{R}^\times : |\mathrm{Nm}(z)| = 1 \}$

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
j(\mathcal{O}_K^\times) \arrow[r, phantom, "\subset"] \arrow[rrr, bend left] & S \arrow[rr] \arrow[d] & & \Gamma \subset H \supset \Pi \arrow[d] \\
 & K_\mathbb{R}^\times \arrow[rr, "\ell"] \arrow[d, "|\mathrm{Nm}|"'] & & (\mathbb{R}^\Sigma)^+ \arrow[d, "\mathrm{Tr}"] \\
 & \mathbb{R}^{>0} \arrow[rr, "\log"] \arrow[d] & & \mathbb{R} \arrow[d] \\
 & 0 & & 0
\end{tikzcd}$$

To show: $S = j(\mathcal{O}_K^\times) \cdot \ell^{-1}(\Pi)$

So for $\rightsquigarrow$ $y \in S$ need $(*)$ $\exists\, u \in \mathcal{O}_K^\times : y \in j(u) \cdot \ell^{-1}(\Pi)$.

Choose $c_\sigma > 0$ with $c_{\bar\sigma} = c_\sigma$ and

$c := \prod_{\sigma \in \Sigma} c_\sigma > \left(\frac{2}{\pi}\right)^s \cdot \sqrt{|d_K|}$

For $y = (y_\sigma)_\sigma \in S$ we have $|y_{\bar\sigma}| = |\overline{y_\sigma}| = |y_\sigma|$

and $\prod_\sigma |y_\sigma| = 1$. Set $c'_\sigma := \frac{c_\sigma}{|y_\sigma|}$

$\Rightarrow \forall \sigma: c'_{\bar\sigma} = c'_\sigma$ and $\prod_\sigma c'_\sigma = c$

Minkowski $\Rightarrow$ $\exists\, a \in \mathcal{O}_K \setminus \{0\}$ with

$\forall \sigma : |\sigma(a)| < c'_\sigma$.

Then $|N_{K/\mathbb{Q}}(a)| = \prod_\sigma |\sigma(a)| < \prod_\sigma c'_\sigma = c$

$\quad\quad \| $

$[\mathcal{O}_K : (a)]$ $\Rightarrow$ only fin. many possibilities for the ideal (a).

[indep. of y].

For any (such) $\mathfrak{a}$ find bounded subset $\Gamma_{\mathfrak{a}}$ and take $\Gamma := \bigcup_{\mathfrak{a}} \Gamma_{\mathfrak{a}}$.

So fix $\mathfrak{a}$. Fix generator b of $\mathfrak{a}$. Write $a = u^{-1} b$ with $u \in \mathcal{O}_K^\times$.

$$\Rightarrow \forall \sigma: \left| \frac{\sigma(b)}{\sigma(u)} \right| = |\sigma(a)| < c'_\sigma = \frac{c_\sigma}{|y_\sigma|}$$

$$\Rightarrow \left| \frac{y_\sigma}{\sigma(u)} \right| < \frac{c_\sigma}{|\sigma(b)|}$$

Let $\boxed{z := \frac{y}{j(u)}}$ $\Rightarrow$ $\boxed{|z_\sigma| = \left|\frac{y_\sigma}{\sigma(u)}\right| < \frac{c_\sigma}{|\sigma(b)|}}$

Also $y \in S \Rightarrow \prod_\sigma |y_\sigma| = 1$

$u \in \mathcal{O}_K^\times \Rightarrow \prod_\sigma |\sigma(u)| = 1$

$\Rightarrow \boxed{\prod_\sigma |z_\sigma| = 1.}$

$\Rightarrow \forall \sigma: |z_\sigma| > \prod_{\sigma' \neq \sigma} \frac{c_{\sigma'}}{|\sigma'(b)|}$

So: $z \in \ell^{-1}(\Gamma)$

$y = j(u) \cdot z$

$\Longrightarrow$ (*).

Let $N := \max\left\{ \frac{c_\sigma}{|\sigma(b)|} \,\middle|\, \sigma \in \Sigma \right\} \cup \left\{ \prod_{\sigma' \neq \sigma} \frac{|\sigma'(b)|}{c_{\sigma'}} \,\middle|\, \sigma \in \Sigma \right\}$

$\Rightarrow \forall \sigma:$ $\boxed{\frac{1}{N} < |z_\sigma| < N.}$ $\Rightarrow -\log N < \log|z_\sigma| < \log N.$

$\Gamma_{\mathfrak{a}} := \{ (t_\sigma)_\sigma \in H \mid \forall \sigma: |t_\sigma| < \log N \}.$ $\Rightarrow \ell(z) \in \Gamma.$

qed.