Correction:

Proposition 6.7.4: In general $\mathrm{diff}_{B/A}$ is the ideal that is generated by $\frac{df}{dX}(\beta)$ for all $\beta \in B$ $\boxed{\text{with } L = K(\beta)}$ and minimal polynomial f over K.

$$\boxed{\mathrm{disc}_{B/A} = \mathrm{Nm}_{L/K}(\mathrm{diff}_{B/A}).}$$

$\left\{ \frac{a}{s} \;\middle|\; \begin{matrix} a \in A \\ s \in S \end{matrix} \right\}$

To prove Proposition 6.8.2 in general use the following facts from commutative algebra:

- For any prime ideal $\mathfrak{p}$ of a ~~ring~~ integral domain A the set $S := A \setminus \mathfrak{p}$ is multiplicative and the ring $A_{\mathfrak{p}} := S^{-1}\mathfrak{p}$ is called the *localization of A at $\mathfrak{p}$*.

- For any ideal $\mathfrak{a} \subset A$ the set $\mathfrak{a}_{\mathfrak{p}} := S^{-1}\mathfrak{a}$ is an ideal of $A_{\mathfrak{p}}$. $\left\{ \frac{a}{s} \;\middle|\; \begin{matrix} a \in \mathfrak{a} \\ s \in S \end{matrix} \right\}$

Now assume that A is Dedekind and that $\mathfrak{p}$ is a maximal ideal.

- Then $A_{\mathfrak{p}}$ is a principal ideal domain. All nonzero ideals are $\mathfrak{p}_{\mathfrak{p}}^n$ for $n \geq 0$. $\mathfrak{p}_{\mathfrak{p}} = \pi \cdot A_{\mathfrak{p}}$ for any $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$.

- For any nonzero ideals $\mathfrak{a}, \mathfrak{a}' \subset A$ we have $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}'_{\mathfrak{p}}$ if and only if the exponents of $\mathfrak{p}$ in the prime factorizations of $\mathfrak{a}$ and $\mathfrak{a}'$ coincide.

Now let B be the integral closure of A in a finite separable extension $L/\operatorname{Quot}(A)$.

- Then $B_{\mathfrak{p}} := S^{-1}B$ is a principal ideal domain.

- The formation of $\mathrm{disc}_{B/A}$ and $\mathrm{diff}_{B/A}$ and the relative ideal norm commutes with localization at $\mathfrak{p}$.

7 Zeta functions

7.1 Riemann zeta function

Definition 7.1.1: The *Riemann zeta function* is defined by the series

$$\boxed{\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.}$$

Proposition 7.1.2: This series converges absolutely and locally uniformly for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a holomorphic function there.

Proof: For $\mathrm{Re}(s) \geq \underset{\text{fix}}{\sigma} > 1$ we have $\sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\mathrm{Re}\, s} \leq \sum_{n=1}^{\infty} n^{-\sigma} < \infty$

q.e.d.

Lemma 7.1.3: For all $\mathrm{Re}(s) > 1$ we have

$$O(1)$$

$$\zeta(s) = \frac{s}{s-1} - s \cdot \int_1^\infty (x - \lfloor x \rfloor) x^{-s-1}\, dx.$$

Proof: $n^{-s} = -x^{-s}\Big|_{x=n}^{x=\infty} = \int_n^\infty s\cdot x^{-s-1}\,dx$

$$\Rightarrow \zeta(s) = \sum_{n=1}^{\infty} \int_n^\infty \underbrace{s\cdot x^{-s-1}}_{|\cdot| \le s\cdot x^{-\sigma-1}}\,dx = \int_1^\infty s\cdot x^{-s-1}\cdot\lfloor x\rfloor\cdot dx = \int_1^\infty s\cdot x^{-s}\,dx + \int_1^\infty s x^{-s-1}\cdot(\lfloor x\rfloor - x)\cdot dx$$

$\mathrm{Re}(s) \ge \sigma > 1$

$$\frac{s\cdot x^{1-s}}{1-s}\Big|_1^\infty = \frac{s}{s-1}$$

qed

Proposition 7.1.4: The function $\zeta(s) - \frac{1}{s-1}$ extends uniquely to a holomorphic function on the region $\mathrm{Re}(s) > 0$.

Proof: $\int_1^\infty \left|(x-\lfloor x\rfloor)\cdot x^{-s-1}\right| dx \le \int_1^\infty x^{-\sigma-1}\,dx = \frac{x^{-\sigma}}{-\sigma}\Big|_1^\infty = \frac{1}{\sigma}$ if $\mathrm{Re}(s) \ge \sigma > 0$

$$\frac{s}{s-1} = \frac{1}{s-1} + 1$$

qed.

Remark 7.1.5: It is known that $\zeta(s)$ extends uniquely to a meromorphic function on $\mathbb{C}$ with a single pole at $s = 1$. This extension is again denoted by $\zeta(s)$.

Throughout the following we use the branch of the logarithm with $\log 1 = 0$.

Proposition 7.1.6: An infinite product of non-zero complex numbers $\prod_{k\geqslant 1} z_k$ converges to a non-zero value if and only if $\lim_{k\to\infty} z_k = 1$ and $\sum_{k\geqslant 1} \log z_k$ converges.

i.e. $\lim_{n\to\infty} \prod_{k=1}^{n} z_k$

Proposition 7.1.7: For all $\mathrm{Re}(s) > 1$ we have the *Euler product*

$$\boxed{\zeta(s) = \prod_{p \text{ prime}} (1-p^{-s})^{-1} \neq 0.}$$

Proof: $\prod_p (1-p^{-s})^{-1} = \prod_p \left(\sum_{\nu_p \geq 0} p^{-\nu_p s}\right) \overset{?}{=} \sum_{\substack{\nu_p \geq 0 \\ \text{for all } p}} \left(\prod_p p^{\nu_p}\right)^{-s} = \sum_{n\geq 1} n^{-s}$ $\quad \boxed{?}$

$\log \prod_p \underbrace{(1-p^{-s})^{-1}}_{\to 1 \text{ for } p\to\infty} = \sum_p \log\left((1-p^{-s})^{-1}\right) = \sum_p O(p^{-s}) < \infty.$

$\underbrace{\prod_{p\leq N} (1-p^{-s})^{-1}}_{} = \ldots = \underbrace{\sum_{\substack{n\geq 1 \text{ such that} \\ \text{all prime factors are } \leq N}} n^{-s}.}_{\to \zeta(s).}$

Let $N \to \infty$

qed.

Proposition 7.1.8: We have

$$\boxed{\sum_{p \text{ prime}} p^{-s} \;=\; \log \tfrac{1}{s-1} + O(1) \quad \text{for real } s \to 1+.}$$

Proof: $\mathrm{Re}(s) \geq 1$

$$\log \zeta(s) = \sum_p \log\left((1-p^{-s})^{-1}\right) = \sum_p \sum_{n\geq 1} \frac{p^{-sn}}{n} = \sum_p p^{-s} + \sum_p \sum_{n\geq 2} \frac{p^{-sn}}{n}.$$

$$= O(1)$$

$$\sum_p \sum_{n\geq 2} \left|\frac{p^{-sn}}{n}\right| \leq \sum_p \sum_{n\geq 2} p^{-n} = \sum_p p^{-2} \cdot \frac{1}{1-p^{-1}} \leq \sum_p 2 \cdot p^{-2} \leq 2\zeta(2) < \infty$$

$$\Rightarrow \sum p^{-s} = O(1) + \log(\zeta(s)) = O(1) + \log\left(\frac{1}{s-1}\right) + O(1).$$

qed.

Definition 7.1.9: For $x \in \mathbb{R}$ we denote the number of primes $\leqslant x$ by $\pi(x)$.

$$\pi(x) = \frac{x}{\log x} \cdot (1 + o(1))$$

Corollary 7.1.10: There is no $\varepsilon > 0$ such that for $x \to \infty$ we have

$$\boxed{\pi(x) \;=\; O\Big(\frac{x}{(\log x)^{1+\varepsilon}}\Big).}$$

In particular there exist infinitely many primes.

Proof: Suppose $\pi(x) \leq \dfrac{C x}{(\log x)^{1+\varepsilon}}$

$$\Rightarrow \sum_p p^{-s} \leq \sum_{n=0}^{\infty} \left(\pi(e^{n+1}) - \pi(e^n)\right) \cdot (e^{-n})^s \leq \sum_{n\geq 0} \frac{Ce^n}{[(n+1)e]^{1+\varepsilon}} \cdot e^{-ns}$$

$$e^n < p \leq e^{n+1} \Rightarrow p^{-s} \leq (e^{-n})^s$$

$$\leq \sum_{n\geq 0} \frac{C}{(n+1)^{1+\varepsilon} \cdot e^{1+\varepsilon}} e^{n(1-s)} \leq \sum_{m\geq 1} \frac{C'}{m^{1+\varepsilon}} < \infty. \ \lightning.$$

qed

7.2 Dedekind zeta function

Fix a number field K of degree n over $\mathbb{Q}$.

Definition 7.2.1: The *Dedekind zeta function of K* is defined by the series

$$\boxed{\zeta_K(s) \;:=\; \sum_{\mathfrak{a}} \mathrm{Nm}(\mathfrak{a})^{-s},}$$

where the sum extends over all non-zero ideals $\mathfrak{a} \subset \mathcal{O}_K$.

Proposition 7.2.2: This series converges absolutely and locally uniformly for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a holomorphic function there, and we have the *Euler product*

$$\boxed{\zeta_K(s) \;=\; \prod_{\mathfrak{p}} \bigl(1 - \mathrm{Nm}(\mathfrak{p})^{-s}\bigr)^{-1} \;\neq\; 0,}$$

extended over all maximal ideals $\mathfrak{p} \subset \mathcal{O}_K$.

Proof: $\log \prod_{\mathfrak{p}} \bigl(1-\mathrm{Nm}(\mathfrak{p})^{-s}\bigr)^{-1} = \sum_{\mathfrak{p}} -\log\bigl(1-\mathrm{Nm}(\mathfrak{p})^{-s}\bigr) = \boxed{\sum_{\mathfrak{p}} \sum_{m\geq 1} \frac{\mathrm{Nm}(\mathfrak{p})^{-sm}}{m}}$

At most n primes $\mathfrak{p}$ over p. $\mathrm{Nm}(\mathfrak{p}) \geq p$

$= \sum_{\substack{\mathbb{Z} \ni p \text{ prime} \\ m \geq 1}} \sum_{\mathfrak{p}|p} \frac{\mathrm{Nm}(\mathfrak{p})^{-sm}}{m} \leq \sum_{p} n \cdot \frac{p^{-sm}}{m}$

converges absolutely, locally uniformly for $\mathrm{Re}(s) > 1$.

$\prod_{p \leq N} \prod_{\mathfrak{p}|p} \bigl(1-\mathrm{Nm}(\mathfrak{p})^{-s}\bigr)^{-1} = \prod_{\substack{p\leq N \\ \mathfrak{p}|p}} \sum_{\nu\geq 0} \mathrm{Nm}(\mathfrak{p})^{-\nu s} = \sum_{\mathfrak{a} \subset \mathcal{O}_K \,:\, \forall \mathfrak{p}|\mathfrak{a}:\ \mathfrak{p}|p\leq N} \mathrm{Nm}(\mathfrak{a})^{-s} \xrightarrow{N\to\infty} \sum_{\mathfrak{a}} \mathrm{Nm}(\mathfrak{a})^{-s} = \zeta_K(s)$

qed.

Proposition 7.2.3: We have

$$\log \zeta_K(s) \;=\; \sum_{\mathfrak{p}} \mathrm{Nm}(\mathfrak{p})^{-s} + \big(\text{holomorphic for } \mathrm{Re}(s) > \tfrac{1}{2}\big).$$

Proof: $\log \zeta_K(s) = \sum_{\mathfrak{p}} \mathrm{Nm}(\mathfrak{p})^{-s} + \underbrace{\sum_{\mathfrak{p}} \sum_{m \geq 2} \frac{\mathrm{Nm}(\mathfrak{p})^{-sm}}{m}}$

$$\leq \sum_{p} n \cdot \sum_{m \geq 2} p^{-\sigma m} = \sum_{p} \frac{n \cdot p^{-2\sigma}}{1 - p^{-\sigma}} \leq 2n \cdot \underbrace{\sum_{p} p^{-2\sigma}}_{\leq \zeta(2\sigma)}$$

qed.

Theorem 7.2.4: The function $\zeta_K(s)$ extends uniquely to a meromorphic function on the region $\mathrm{Re}(s) > 1 - \frac{1}{n}$ which is holomorphic except for a pole of order 1 at $s = 1$.

Proposition 7.2.5: We have

$$\boxed{\sum_{\mathfrak{p}} \mathrm{Nm}(\mathfrak{p})^{-s} \;=\; \log \tfrac{1}{s-1} + O(1) \quad \text{for real } s \to 1+.}$$

Corollary 7.2.6: There exist infinitely many rational primes that split totally in $\mathcal{O}_K$.

Proof: Let $\tilde{L}$ be the galois closure of L over $\mathbb{Q}$,

$\Rightarrow \exists \infty$ many $\tilde{\mathfrak{q}} < \mathcal{O}_{\tilde{L}}$ with $\tilde{\mathfrak{q}} \mid p \in \mathbb{Z}$ and $\mathrm{Nm}(\tilde{\mathfrak{q}}) = p$.

$\exists$ only fin. many ramified primes.

$\tilde{\mathfrak{q}}$ unramified over p, $k(\tilde{\mathfrak{q}}) = \mathbb{F}_p$ $\Rightarrow$ p totally split in $\tilde{L}$.

$\Rightarrow$ " " " " L. qed.

$1 \to \{\pm 1\}^r \times (S^1)^s \to K_{\mathbb{R}}^\times \xrightarrow{\log} K_{\mathbb{R}} \to 0$

7.3 Analytic class number formula

As before we set $\Sigma := \mathrm{Hom}(K, \mathbb{C})$ and let r be the number of embeddings $K \hookrightarrow \mathbb{R}$ and s the number of pairs of complex conjugate non-real embeddings $K \hookrightarrow \mathbb{C}$. With $K_{\mathbb{C}} := \mathbb{C}^\Sigma$ and

$$K_{\mathbb{R}} \;:=\; \{(z_\sigma)_\sigma \in K_{\mathbb{C}} \;\big|\; \forall \sigma \in \Sigma\colon z_{\bar\sigma} = \bar z_\sigma\}$$

as in §3.4 we then have

$$K_{\mathbb{R}} \cap \mathbb{R}^\Sigma \;=\; \{(t_\sigma)_\sigma \in \mathbb{R}^\Sigma \;\big|\; \forall \sigma \in \Sigma\colon t_{\bar\sigma} = t_\sigma\}.$$

The $\mathbb{R}$-subspace

$$H := \ker\big(\mathrm{Tr}\colon K_{\mathbb{R}} \cap \mathbb{R}^\Sigma \to \mathbb{R}\big)$$

from §5.2 therefore becomes a euclidean vector space by its embedding $H \subset K_{\mathbb{R}} \subset K_{\mathbb{C}}$ and the scalar product from §4.1. By §2.2 it is thus endowed with a canonical translation invariant measure $d\,\mathrm{vol}$. Recall from Theorem 5.3.1 that $\Gamma := \ell(j(\mathcal{O}_K^\times))$ is a complete lattice in H.

Definition 7.3.1: The *regulator of K* is the real number

$$R \;:=\; \mathrm{vol}(H/\Gamma) \;>\; 0.$$

Let $w := |\mu(K)|$ denote the number of roots of unity in K and let $h := |\,\mathrm{Cl}(\mathcal{O}_K)|$ the class number.

Theorem 7.2.7: *Analytic class number formula:* The residue of $\zeta_K(s)$ at $s = 1$ is

$$\underline{\mathrm{Res}_{s=1}\,\zeta_K(s)} \;=\; \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} \;>\; 0.$$