Reminder:

Fix a number field K of degree n over $\mathbb{Q}$.

Definition 7.2.1: The *Dedekind zeta function of K* is defined by the series

$$\boxed{\zeta_K(s) \ := \ \sum_{\mathfrak{a}} \mathrm{Nm}(\mathfrak{a})^{-s},}$$

$\mathrm{Re}(s) > 1.$

where the sum extends over all non-zero ideals $\mathfrak{a} \subset \mathcal{O}_K$.

We want to prove:

Theorem 7.2.4: The function $\zeta_K(s)$ extends uniquely to a meromorphic function on the region $\mathrm{Re}(s) > 1 - \frac{1}{n}$ which is holomorphic except for a pole of order 1 at $s = 1$.

Theorem 7.2.7: *Analytic class number formula:* The residue of $\zeta_K(s)$ at $s = 1$ is

$$\boxed{\mathrm{Res}_{s=1}\,\zeta_K(s) \ = \ \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} \ > \ 0.}$$

Here r is the number of embeddings $K \hookrightarrow \mathbb{R}$ and s the number of pairs of complex conjugate non-real embeddings $K \hookrightarrow \mathbb{C}$. Moreover $w := |\mu(K)|$ denotes the number of roots of unity in K and $h := |\,\mathrm{Cl}(\mathcal{O}_K)|$ the class number. The *regulator of K* is the real number $R \ := \ \mathrm{vol}(H/\Gamma) \ > \ 0.$

$\cong K \otimes_{\mathbb{Q}} \mathbb{C}$ $\quad |\Sigma| = r+2s$

As before we set $\Sigma := \mathrm{Hom}(K, \mathbb{C})$. With $K_{\mathbb{C}} := \mathbb{C}^\Sigma$ and

$$\mathbb{R}^\Sigma \neq \quad K_{\mathbb{R}} := \{(z_\sigma)_\sigma \in K_{\mathbb{C}} \mid \forall \sigma \in \Sigma\colon z_{\bar\sigma} = \bar z_\sigma\} \quad \cong K \otimes_{\mathbb{Q}} \mathbb{R}$$

as in §3.4 we then have

$$K_{\mathbb{R}} \cap \mathbb{R}^\Sigma = \{(t_\sigma)_\sigma \in \mathbb{R}^\Sigma \mid \forall \sigma \in \Sigma\colon t_{\bar\sigma} = t_\sigma\}. \quad \cong \mathbb{R}^{r+s}$$

The $\mathbb{R}$-subspace

$$\mathrm{Tr}((t_\sigma)) = \sum_\sigma t_\sigma \, .$$

$$H := \ker\big(\mathrm{Tr}\colon K_{\mathbb{R}} \cap \mathbb{R}^\Sigma \to \mathbb{R}\big)$$

from §5.2 therefore becomes a euclidean vector space by its embedding $H \subset K_{\mathbb{R}} \subset K_{\mathbb{C}}$ and the scalar product from §4.1. By §2.2 it is thus endowed with a canonical translation invariant measure $d\,\mathrm{vol}$. Recall from Theorem 5.3.1 that $\Gamma := \ell(j(\mathcal{O}_K^\times))$ is a complete lattice in H.

Proof of Thm: (1)

$$\zeta_K(s) = \sum_{0\neq \mathfrak{a} \subset \mathcal{O}_K} (Nm(\mathfrak{a}))^{-s} = \sum_{i=1}^{h} \sum_{\substack{0\neq \mathfrak{a} \subset \mathcal{O}_K \\ [\mathfrak{a}]=[\mathfrak{a}_i^{-1}]}} (Nm(\mathfrak{a}))^{-s}$$

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ be rep's of $Cl(\mathcal{O}_K)$

$$[\mathfrak{a}] = [\mathfrak{a}_i^{-1}] \iff \mathfrak{a} = x\mathfrak{a}_i^{-1} \text{ for some } x \in K^\times$$

$$\iff x \in \mathfrak{a}\mathfrak{a}_i \quad \text{" "}$$

$$\mathfrak{a} \subset \mathcal{O}_K \rightsquigarrow \underbrace{x \in \mathfrak{a}_i \setminus \{0\}}$$

$$x\mathfrak{a}_i^{-1} = x'\mathfrak{a}_i^{-1} \iff x', x \text{ differ by } \underbrace{\mathcal{O}_K^\times}.$$

$$Nm(x\mathfrak{a}_i^{-1}) = \frac{Nm(x)}{Nm(\mathfrak{a}_i)}$$

$$= \sum_{i=1}^{h} \sum_{\substack{x \in \mathfrak{a}_i \setminus \{0\} \\ \text{modulo } \mathcal{O}_K^\times}} \left(\frac{Nm(x)}{Nm(\mathfrak{a}_i)}\right)^{-s}$$

$$= \sum_{i=1}^{h} Nm(\mathfrak{a}_i)^{s} \cdot \boxed{\sum_{\substack{x \in \mathfrak{a}_i \setminus \{0\} \\ \text{mod } \mathcal{O}_K^\times}} Nm(x)^{-s}}$$

$$\parallel$$

$$\zeta_{\mathfrak{a}_i}(s)$$

For any $0 \neq \mathfrak{a}$ fractional ideal

$$\zeta_{\mathfrak{a}}(s) := \sum_{\substack{x \in \mathfrak{a} \setminus \{0\} \\ \text{mod } \mathcal{O}_K^\times}} Nm(x)^{-s}$$

(2) Need to estimate for any $m \geq 1$:

$$\boxed{d_{\mathfrak{a}}(m) := \#\left(\{ x \in \mathfrak{a} \setminus \{0\} \mid Nm(x) \leq m \} \text{ modulo } \mathcal{O}_K^\times \right)}$$

(3) $K \xhookrightarrow{j} K_{\mathbb{R}}\,, \quad x \mapsto (\sigma(x))_\sigma$

$\mathfrak{a} \longrightarrow j(\mathfrak{a}) \subseteq K_{\mathbb{R}}$

$K^\times \xhookrightarrow{j} K_{\mathbb{R}}^\times \xrightarrow{\ell} K_{\mathbb{R}} \cap \mathbb{R}^\Sigma =: (\mathbb{R}^\Sigma)^+$

$(z_\sigma)_\sigma \longmapsto (\log|z_\sigma|)_\sigma$

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
\tilde{\Gamma} \subset \mathcal{O}_K^\times \arrow[r, hook, "j"] \arrow[d] \arrow[rrr, bend left, "\ell(j(\mathcal{O}_K^\times))=\Gamma"] & S \arrow[r, two heads, "\ell"] \arrow[d] \arrow[dr, phantom, "///"] & H \arrow[d] \\
K^\times \arrow[r, hook, "j"] \arrow[ur, phantom, "///"] \arrow[dr, "|N_{K/\mathbb{Q}}|"'] & K_{\mathbb{R}}^\times \arrow[r, two heads, "\ell"] \arrow[d, "|N_m|"] \arrow[dr, phantom, "///"] & (\mathbb{R}^\Sigma)^+ \arrow[d, "\mathrm{Tr}"] \\
 & \mathbb{R}^{>0} \arrow[r, "\log", "\sim"'] \arrow[d] & \mathbb{R} \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^{>0} \arrow[r, hook] \arrow[dr, "n"] & K_{\mathbb{R}}^\times \arrow[d, "N_m"] \\
 & \mathbb{R}^{>0}
\end{tikzcd}$$

$t \mapsto t$, $\quad t \mapsto t^n$

(4) Write $\mathcal{O}_K^\times = \mu(K) \times \tilde{\Gamma}$, $\tilde{\Gamma} = \prod_{i=1}^{r+s-1} u_i^{\mathbb{Z}}$.

With $\gamma_i := \ell(j(u_i))$ we set $\Gamma = \bigoplus_{i=1}^{r+s-1} \mathbb{Z}\gamma_i$.

$\Phi := \{\sum \lambda_i \gamma_i \mid \forall i: \lambda_i \in [0,1[\}$

fundamental domain for $\Gamma \subset H$.

i.e. $\Phi \times \Gamma \xrightarrow{\text{bij.}} H$

$(z, \gamma) \mapsto z + \gamma$

$\Rightarrow \quad \mathbb{R}^{>0} \times \ell^{-1}(\Phi) \times j(\tilde{\Gamma}) \xrightarrow{\text{bij.}} K_{\mathbb{R}}^\times\,,$

$(t, s, u) \longmapsto tsu\,.$

$\Rightarrow \ell^{-1}(\Phi) \times j(\tilde{\Gamma}) \xrightarrow{\text{bij.}} S$

$$Nm(t\cdot u) = t^n$$

$$\Rightarrow Nm(t u) \le m \iff t^n \le m \iff t \le \sqrt[n]{m}.$$

(J) $$\Rightarrow d_n(m) = \#\{x \in (\mathcal{O}\setminus\{0\})/\mathcal{O}_K^\times \mid |Nm(x)| \le m\}$$

$$= \frac{1}{w}\cdot \#\{x \in (\mathcal{O}\setminus\{0\})/\tilde{\Gamma} \mid |Nm(x)| \le m\}.$$

$$= \frac{1}{w}\cdot \#\{z \in (j(\mathcal{O}) \cap K_{\mathbb{R}}^\times)/j(\tilde{\Gamma}) \mid Nm(z) \le m\}$$

$$= \frac{1}{w}\cdot \#\{z \in j(\mathcal{O}) \cap \mathbb{R}^{>0}\cdot \ell^{-1}(\Phi) \mid Nm(z) \le m\}$$

$$= \frac{1}{w}\cdot \#\{z \in j(\mathcal{O}) \cap\,]0, \sqrt[n]{m}]\cdot \ell^{-1}(\Phi)\}$$

$$= \frac{1}{w}\cdot \#(j(\mathcal{O}) \cap X_{\sqrt[n]{m}}) \quad \text{for} \quad \boxed{X_c :=]0,c]\cdot \ell^{-1}(\Phi)}$$

for any $c \in \mathbb{R}^{>0}$.

Next steps: Estimate the number of points in a lattice and a convex subset.