

Reminder:

Fix a number field K of degree n over \mathbb{Q} .

Definition 7.2.1: The *Dedekind zeta function* of K is defined by the series

$$\zeta_K(s) := \sum_{\mathfrak{a}} \text{Nm}(\mathfrak{a})^{-s},$$

$\text{Re}(s) > 1.$

where the sum extends over all non-zero ideals $\mathfrak{a} \subset \mathcal{O}_K$.

We want to prove:

Theorem 7.2.4: The function $\zeta_K(s)$ extends uniquely to a meromorphic function on the region $\text{Re}(s) > 1 - \frac{1}{n}$ which is holomorphic except for a pole of order 1 at $s = 1$.

Theorem 7.2.7: *Analytic class number formula:* The residue of $\zeta_K(s)$ at $s = 1$ is

$$\text{Res}_{s=1} \zeta_K(s) = \frac{2^r (2\pi)^s R h}{w \sqrt{|d_K|}} > 0.$$

Here r is the number of embeddings $K \hookrightarrow \mathbb{R}$ and s the number of pairs of complex conjugate non-real embeddings $K \hookrightarrow \mathbb{C}$. Moreover $w := |\mu(K)|$ denotes the number of roots of unity in K and $h := |\text{Cl}(\mathcal{O}_K)|$ the class number. The regulator of K is the real number $R := \text{vol}(H/\Gamma) > 0$.

$$|\Sigma| = r + 2s$$

As before we set $\Sigma := \text{Hom}(K, \mathbb{C})$. With $K_{\mathbb{C}} := \mathbb{C}^{\Sigma}$ and

$$\mathbb{R}^{\Sigma} \neq K_{\mathbb{R}} := \{(z_{\sigma})_{\sigma} \in K_{\mathbb{C}} \mid \forall \sigma \in \Sigma: z_{\bar{\sigma}} = \bar{z}_{\sigma}\} \cong K_{\mathbb{R}}$$

as in §3.4 we then have

$$K_{\mathbb{R}} \cap \mathbb{R}^{\Sigma} = \{(t_{\sigma})_{\sigma} \in \mathbb{R}^{\Sigma} \mid \forall \sigma \in \Sigma: t_{\bar{\sigma}} = t_{\sigma}\} \cong \mathbb{R}^{r+s}$$

The \mathbb{R} -subspace

$$H := \ker(\text{Tr}: K_{\mathbb{R}} \cap \mathbb{R}^{\Sigma} \rightarrow \mathbb{R})$$

$$\text{Tr}(\langle t_{\sigma} \rangle) = \sum_{\sigma} t_{\sigma}$$

from §5.2 therefore becomes a euclidean vector space by its embedding $H \subset K_{\mathbb{R}} \subset K_{\mathbb{C}}$ and the scalar product from §4.1. By §2.2 it is thus endowed with a canonical translation invariant measure $d \text{vol}$. Recall from Theorem 5.3.1 that $\Gamma := \ell(j(\mathcal{O}_K^{\times}))$ is a complete lattice in H .

Proof of Thm. 1) $J_K(s) = \sum_{0 \neq \mathfrak{a} \subset \mathfrak{O}_K} (\text{Nm}(\mathfrak{a}))^{-s} = \sum_{i=1}^h \sum_{\substack{0 \neq \mathfrak{a} \subset \mathfrak{O}_K \\ [\mathfrak{a}] = [\mathfrak{v}_i^{-1}]}} (\text{Nm}(\mathfrak{a}))^{-s}$

Let $\mathfrak{v}_1, \dots, \mathfrak{v}_h$ be reps of \mathfrak{O}_K

$[\mathfrak{a}] = [\mathfrak{v}_i^{-1}] \Leftrightarrow \mathfrak{a} = x \mathfrak{v}_i^{-1}$ for some $x \in K^\times$
 $\Leftrightarrow x \in \mathfrak{a} \mathfrak{v}_i$

$\mathfrak{a} \subset \mathfrak{O}_K \rightsquigarrow x \in \mathfrak{v}_i \setminus \{0\}$

$x \mathfrak{v}_i^{-1} = x' \mathfrak{v}_i^{-1} \Leftrightarrow x', x$ differ by \mathfrak{O}_K^\times

$\text{Nm}(x \mathfrak{v}_i^{-1}) = \frac{\text{Nm}(x)}{\text{Nm}(\mathfrak{v}_i)}$

$= \sum_{i=1}^h \sum_{\substack{x \in \mathfrak{v}_i \setminus \{0\} \\ \text{mod } \mathfrak{O}_K^\times}} \left(\frac{\text{Nm}(x)}{\text{Nm}(\mathfrak{v}_i)} \right)^{-s}$
 $= \sum_{i=1}^h \text{Nm}(\mathfrak{v}_i)^s \cdot \sum_{\substack{x \in \mathfrak{v}_i \setminus \{0\} \\ \text{mod } \mathfrak{O}_K^\times}} \text{Nm}(x)^{-s}$

$\sum_{\substack{x \in \mathfrak{v}_i \setminus \{0\} \\ \text{mod } \mathfrak{O}_K^\times}} \text{Nm}(x)^{-s}$

For any $0 \neq \mathfrak{a}$ fractional ideal
 $J_{\mathfrak{a}}(s) := \sum_{\substack{x \in \mathfrak{a} \setminus \{0\} \\ \text{mod } \mathfrak{O}_K^\times}} \text{Nm}(x)^{-s}$

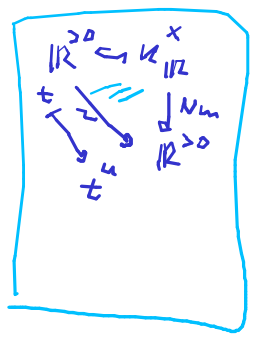
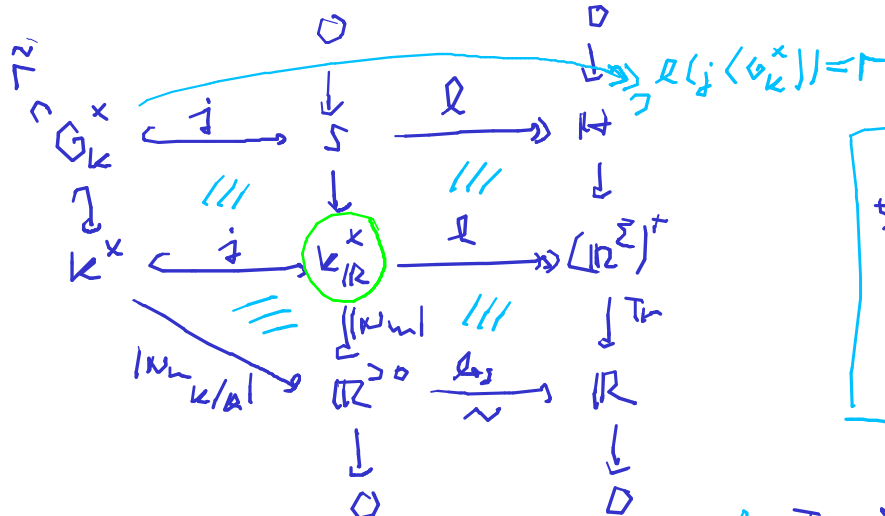
$J_{\mathfrak{v}_i}(s)$

2) Needs estimate for any $m \geq 1$:

$d_{\mathfrak{v}_i}(m) := \#\left\{ \begin{array}{l} x \in \mathfrak{v}_i \setminus \{0\} \\ \text{Nm}(x) \leq m \\ \text{mod } \mathfrak{O}_K^\times \end{array} \right\}$

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$$\begin{array}{c}
 K \xrightarrow{j} K_{\mathbb{R}} \quad , \quad x \mapsto (\sqrt{|x|})/\sigma \\
 \downarrow \quad \downarrow j/\sigma \\
 K^x \xrightarrow{j} K_{\mathbb{R}}^x \xrightarrow{\sigma} K_{(\mathbb{R} \cap \mathbb{R}^{\Sigma})} =: (\mathbb{R}^{\Sigma})^+ \\
 \downarrow \quad \downarrow \\
 \langle \tau \rangle_{\sigma} \mapsto \langle \sigma_j | \tau \rangle_{\sigma}
 \end{array}$$



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Wie $\mathbb{Q}_K^x = \mu(K) \times \mathbb{R}^2$,
 $\mathbb{R}^2 = \prod_{i=1}^{r+s} u_i \mathbb{R}$
 with $\tau_i := \sigma(j(u_i))$
 we set $\Gamma = \bigoplus_{i=1}^{r+s} \mathbb{Z} \tau_i$.

$\Phi := \{ \sum \lambda_i \tau_i \mid \forall i: \tau_i \in [\sigma_j] \}$
fundamental domain for $\Gamma \subset \mathbb{H}$
 i.e. $\Phi \times \Gamma \xrightarrow{\text{id}} \mathbb{H}$
 $(z, \tau) \mapsto z + \tau$

$$\Rightarrow \mathbb{R}^{>0} \times \mathbb{R}^2 \xrightarrow{\text{id}} K_{\mathbb{R}}^x, \quad (t, s, u) \mapsto t \sigma u$$

$$\Rightarrow \mathbb{R}^1(\Phi) \times \mathbb{R}^2 \xrightarrow{\text{id}} \mathbb{H}$$

$$N_m(t) = t^m$$

$$\Rightarrow \underline{N_m(t) \leq m} \Leftrightarrow t^m \leq m \Leftrightarrow \underline{t \leq \sqrt[m]{m}}.$$

⑤

$$\begin{aligned} \Rightarrow d_m(m) &= \#\{x \in (U \setminus \{0\}) / \mathbb{G}_k^X \mid |N_m(x)| \leq m\} \\ &= \frac{1}{\omega} \cdot \#\{x \in (U \setminus \{0\}) / \tilde{\Gamma} \mid |N_m(x)| \leq m\}. \\ &= \frac{1}{\omega} \cdot \#\{z \in (j(U) \cap \mathbb{R}^k) / j(\tilde{\Gamma}) \mid |N_m(z)| \leq m\} \\ &= \frac{1}{\omega} \cdot \#\{z \in j(U) \cap \underline{\mathbb{R}^0} \cdot \bar{e}'(\Phi) \mid \underline{N_m(z)} \leq m\} \\ &= \frac{1}{\omega} \cdot \#\{z \in j(U) \cap]0, \sqrt[m]{m}]. \bar{e}'(\Phi)\} \\ &= \frac{1}{\omega} \cdot \#\langle j(U) \cap X_{\sqrt[m]{m}} \rangle \quad \text{for } X_c :=]0, c]. \bar{e}'(\Phi) \\ &\quad \text{for } c \in \mathbb{R}^0. \end{aligned}$$

Next steps: Estimate the number of points in a lattice and a convex subset.