Reminder:

Fix a number field K of degree n over $\mathbb{Q}$.

Definition 7.2.1: The *Dedekind zeta function of* K is defined by the series

$$\boxed{\zeta_K(s) \;:=\; \sum_{\mathfrak{a}} \mathrm{Nm}(\mathfrak{a})^{-s},}$$

$\mathrm{Re}(s) > 1.$

where the sum extends over all non-zero ideals $\mathfrak{a} \subset \mathcal{O}_K$.

We want to prove:

Theorem 7.2.4: The function $\zeta_K(s)$ extends uniquely to a meromorphic function on the region $\mathrm{Re}(s) > 1 - \frac{1}{n}$ which is holomorphic except for a pole of order 1 at $s = 1$.

Theorem 7.2.7: *Analytic class number formula:* The residue of $\zeta_K(s)$ at $s = 1$ is

$$\boxed{\mathrm{Res}_{s=1}\,\zeta_K(s) \;=\; \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} \;>\; 0.}$$

Here r is the number of embeddings $K \hookrightarrow \mathbb{R}$ and s the number of pairs of complex conjugate non-real embeddings $K \hookrightarrow \mathbb{C}$. Moreover $w := |\mu(K)|$ denotes the number of roots of unity in K and $h := |\,\mathrm{Cl}(\mathcal{O}_K)|$ the class number. The *regulator of* K is the real number $R \;:=\; \mathrm{vol}(H/\Gamma) \;>\; 0.$

$\cong K \otimes_{\mathbb{Q}} \mathbb{C}$ $\qquad |\Sigma| = r+2s$

As before we set $\Sigma := \mathrm{Hom}(K, \mathbb{C})$. With $K_{\mathbb{C}} := \mathbb{C}^{\Sigma}$ and

$$\mathbb{R}^{\Sigma} \neq K_{\mathbb{R}} := \{(z_\sigma)_\sigma \in K_{\mathbb{C}} \mid \forall \sigma \in \Sigma\colon z_{\bar\sigma} = \bar z_\sigma\} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$$

as in §3.4 we then have

$$K_{\mathbb{R}} \cap \mathbb{R}^{\Sigma} = \{(t_\sigma)_\sigma \in \mathbb{R}^{\Sigma} \mid \forall \sigma \in \Sigma\colon t_{\bar\sigma} = t_\sigma\}. \cong \mathbb{R}^{r+s}$$

The $\mathbb{R}$-subspace

$$H := \ker\big(\mathrm{Tr}\colon K_{\mathbb{R}} \cap \mathbb{R}^{\Sigma} \to \mathbb{R}\big) \qquad \mathrm{Tr}((t_\sigma)) = \sum_\sigma t_\sigma .$$

from §5.2 therefore becomes a euclidean vector space by its embedding $H \subset K_{\mathbb{R}} \subset K_{\mathbb{C}}$ and the scalar product from §4.1. By §2.2 it is thus endowed with a canonical translation invariant measure $d\,\mathrm{vol}$. Recall from Theorem 5.3.1 that $\Gamma := \ell(j(\mathcal{O}_K^\times))$ is a complete lattice in H.

Proof of Thm: (1) $J_K(s) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_K} (Nm(\mathfrak{a}))^{-s} = \sum_{i=1}^{h} \sum_{\substack{0 \neq \mathfrak{a} \subset \mathcal{O}_K \\ [\mathfrak{a}] = [\mathfrak{a}_i^{-1}]}} (Nm(\mathfrak{a}))^{-s}$

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be rep's of $Cl(\mathcal{O}_K)$

$[\mathfrak{a}] = [\mathfrak{a}_i^{-1}] \iff \mathfrak{a} = x\mathfrak{a}_i^{-1}$ for some $x \in K^\times$

$\iff x \in \mathfrak{a}\mathfrak{a}_i$ " "

$\mathfrak{a} \subset \mathcal{O}_K \rightsquigarrow \underbrace{x \in \mathfrak{a}_i \setminus \{0\}}$

$x\mathfrak{a}_i^{-1} = x'\mathfrak{a}_i^{-1} \iff x', x$ differ by $\underbrace{\mathcal{O}_K^\times}$.

$Nm(x\mathfrak{a}_i^{-1}) = \dfrac{Nm(x)}{Nm(\mathfrak{a}_i)}$

$$= \sum_{i=1}^{h} \sum_{\substack{x \in \mathfrak{a}_i \setminus \{0\} \\ \text{modulo } \mathcal{O}_K^\times}} \left(\frac{Nm(x)}{Nm(\mathfrak{a}_i)}\right)^{-s}$$

$$= \sum_{i=1}^{h} Nm(\mathfrak{a}_i)^{s} \cdot \boxed{\sum_{\substack{x \in \mathfrak{a}_i \setminus \{0\} \\ \text{mod } \mathcal{O}_K^\times}} Nm(x)^{-s}}$$

$\| \quad J_{\mathfrak{a}_i}(s)$

For any $0 \neq \mathfrak{a}$ fractional ideal

$$J_{\mathfrak{a}}(s) := \sum_{\substack{x \in \mathfrak{a} \setminus \{0\} \\ \text{mod } \mathcal{O}_K^\times}} Nm(x)^{-s}$$

(2) Need to estimate for any $m \geq 1$:

$$\boxed{d_{\mathfrak{a}}(m) := \#\left(\left\{ x \in \mathfrak{a} \setminus \{0\} \,\middle|\, Nm(x) \leq m \right\} \text{ modulo } \mathcal{O}_K^\times\right)}$$

(3) $K \xhookrightarrow{j} K_{\mathbb{R}}$, $x \mapsto (\sigma(x))_\sigma$

$\mathcal{O} \longrightarrow j(\mathcal{O})$

$K^\times \xhookrightarrow{j} K_{\mathbb{R}}^\times \xrightarrow{\ell} K_{\mathbb{R}} \cap \mathbb{R}^\Sigma =: (\mathbb{R}^\Sigma)^+$

$(z_\sigma)_\sigma \longmapsto (\log|z_\sigma|)_\sigma$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
\tilde\Gamma \subset \mathcal{O}_K^\times \arrow[r, hook, "j"] \arrow[d, hook] \arrow[rr, bend left, "\ell(j(\mathcal{O}_K^\times)) = \Gamma"] & & S \arrow[r, two heads, "\ell"] \arrow[d] & H \arrow[d] \\
K^\times \arrow[rr, hook, "j"] \arrow[drr, "|N_{K/\mathbb{Q}}|"'] & & K_{\mathbb{R}}^\times \arrow[r, two heads, "\ell"] \arrow[d, "|N_m|"] & (\mathbb{R}^\Sigma)^+ \arrow[d, "\mathrm{Tr}"] \\
 & & \mathbb{R}^{>0} \arrow[r, "\log", "\sim"'] \arrow[d] & \mathbb{R} \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^{>0} \arrow[r, hook] \arrow[dr] & K_{\mathbb{R}}^\times \arrow[d, "N_m"] \\
 & \mathbb{R}^{>0}
\end{tikzcd}$$

$t \mapsto t^u$

(4) Write $\mathcal{O}_K^\times = \mu(K) \times \tilde\Gamma$, $\tilde\Gamma = \prod_{i=1}^{r+s-1} u_i^{\mathbb{Z}}$.

With $\gamma_i := \ell(j(u_i))$ we set $\Gamma = \bigoplus_{i=1}^{r+s-1} \mathbb{Z}\gamma_i$.

$\Phi := \{\sum \lambda_i \gamma_i \mid \forall i: \lambda_i \in [0,1[\}$ fundamental domain for $\Gamma \subset H$.

i.e. $\Phi \times \Gamma \xrightarrow{bij.} H$, $(z,\gamma) \mapsto z+\gamma$

$\Rightarrow \mathbb{R}^{>0} \times \ell^{-1}(\Phi) \times j(\tilde\Gamma) \xrightarrow{bij.} K_{\mathbb{R}}^\times$, $(t, s, u) \longmapsto tsu$.

$\Rightarrow \ell^{-1}(\Phi) \times j(\tilde\Gamma) \xrightarrow{bij.} S$

$$\mathrm{Nm}(t\cdot u) = t^n$$

$$\Rightarrow \mathrm{Nm}(t\cdot u) \le m \iff t^n \le m \iff t \le \sqrt[n]{m}.$$

$$\Rightarrow d_n(m) = \#\{x \in (\mathcal{O}\setminus\{0\})/\mathcal{O}_K^\times \mid |\mathrm{Nm}(x)| \le m\}$$

$$= \frac{1}{w}\cdot \#\{x \in (\mathcal{O}\setminus\{0\})/\tilde{F} \mid |\mathrm{Nm}(x)| \le m\}.$$

$$= \frac{1}{w}\cdot \#\{z \in (j(\mathcal{O}) \cap K_{\mathbb{R}}^\times)/j(\tilde{F}) \mid \mathrm{Nm}(z) \le m\}$$

$$= \frac{1}{w}\cdot \#\{z \in j(\mathcal{O}) \cap \mathbb{R}^{>0}\cdot \ell^{-1}(\Phi) \mid \mathrm{Nm}(z) \le m\}$$

$$= \frac{1}{w}\cdot \#\{z \in j(\mathcal{O}) \cap \,]0,\sqrt[n]{m}]\cdot \ell^{-1}(\Phi)\}$$

$$= \frac{1}{w}\cdot \#(j(\mathcal{O}) \cap X_{\sqrt[n]{m}}) \quad \text{for } \boxed{X_c := \,]0,c]\cdot \ell^{-1}(\Phi)}$$

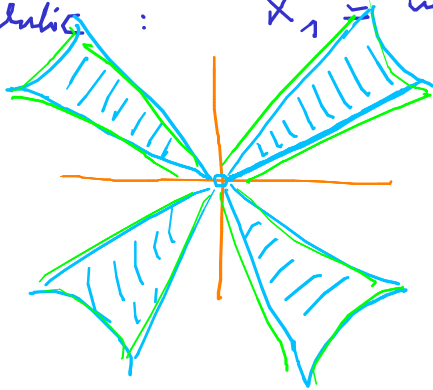
for any $c \in \mathbb{R}^{>0}$.

Next steps: Estimate the number of points in a lattice and a ~~[illegible]~~ subset.

⑥ Note: $X_c = c \cdot X_1$

⑦ Example: K imag. quadratic: X_1 = unit circle $\setminus \{0\}$

K real quadratic: $X_1 =$



⑧ Claim: X_1 is bounded.

Because: $\overline{\Phi} \subset H$ is compact,

$$K_{\mathbb{R}}^{\times} \xrightarrow{\sim} \overbrace{\{\pm 1\}^r \times (S^1)^s}^{\text{compact}} \times (\mathbb{R}^{\Sigma})^+ \quad \text{homeomorphism.}$$

$$(z_\sigma)_\sigma \longmapsto \left(\left(\frac{z_{\sigma_i}}{|z_{\sigma_i}|} \right)_i , \ \ell(z) \right)$$

$\Rightarrow$ $\ell^{-1}(\overline{\Phi})$ compact $\Rightarrow$ bounded $\Rightarrow$ $X_1 \subset \,]0,1] \cdot \ell^{-1}(\overline{\Phi})$ = bounded.

qed

(9) ∂K_1 is Lipschitz parametrizable of dimension $n-1$.

Because: $\partial\Phi$ = finite union of polytopes of dim. $r+s-2$.

ℓ analytic $\Rightarrow$ $\ell^{-1}(\partial\Phi)$ = Lipschitz parametrizable of dim. $r+2s-2=n-2$.

$$\Rightarrow \partial X_1 = \underbrace{[0,1]\cdot\ell^{-1}(\partial\Phi)}_{n-1} \cup \underbrace{\ell^{-1}(\Phi)}_{n-1}.$$

(10) **Theorem:** Let L be a complete lattice in a n-dim real vector space V of dim. $n<\infty$. Let $X\subset V$ be a bounded subset such that ∂X is Lip. parametrizable of dim. $n-1$. Then $\forall\, c>0$:

$$\#(L\cap cX) = \frac{\mathrm{vol}(X)}{\mathrm{vol}(V/L)}\cdot c^n + O(c^{n-1}). \text{ for } c\to\infty.$$

See [Lang: Alg. Number Theory Ch. VI §2.]

(11) Write each $\gamma_j = (\gamma_{j,\sigma})_{\sigma\in\Sigma}$

Recall $\Sigma = \mathrm{Hom}(K,\mathbb{C}) = \{\sigma_1, \dots, \sigma_r, \sigma_{r+1}, \dots, \sigma_{r+s}, \overline{\sigma_{r+1}}, \dots, \overline{\sigma_{r+s}}\}$

For each j set $f_j := \begin{cases} 1 & \text{if } j \le r \iff \sigma_j \text{ real.} \\ 2 & \text{if } j > r \iff \sigma_j \text{ complex} \end{cases}$.

$$R := \left| \det \left(f_j \cdot \gamma_{j,\sigma_i} \right)_{i,j=1..r+s-1} \right| .$$

= covolume of Γ = regulator.

(12) Compute: $\left| \det \begin{pmatrix} 1 & \gamma_{1\sigma_1} & \cdots & \gamma_{r+s-1,\sigma_1} \\ \vdots & \vdots & & \vdots \\ 1 & \gamma_{1,\sigma_{r+s}} & \cdots & \gamma_{r+s-1,\sigma_{r+s}} \end{pmatrix} \right|$

$$= 2^{-s} \cdot \left| \det \begin{pmatrix} f_1 & f_1\gamma_{1,\sigma_1} & \cdots & f_1 \cdot \gamma_{r+s-1,\sigma_1} \\ \vdots & \vdots & & \vdots \\ f_{r+s} & f_{r+s}\gamma_{1,\sigma_{r+s}} & \cdots & f_{r+s}\gamma_{r+s-1,\sigma_{r+s}} \end{pmatrix} \right|$$

add all rows to the last

$$= 2^{-s} \cdot \left| \det \begin{pmatrix} f_1 & f_1\gamma_{1,\sigma_1} & \cdots \\ \vdots & & \\ f_{r+s-1} & f_{r+s-1}\gamma_{1\sigma_{r+s-1}} & \cdots \\ n & 0 \cdots & 0 \end{pmatrix} \right| = 2^{-s} \cdot n \cdot R.$$

(13) $K_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n, \quad (z_\sigma)_\sigma \longmapsto (z_{\sigma_1}, \dots, z_{\sigma_r}, \operatorname{Re} z_{\sigma_{r+1}}, \operatorname{Im} z_{\sigma_{r+1}}, \dots, \dots, \operatorname{Im} z_{\sigma_{r+s}})$

$$\begin{array}{ccc} K_{\mathbb{R}} & \xrightarrow{\sim} & \mathbb{R}^n \\ \cup & & \cup \\ X_1 & \xrightarrow{\sim} & Y_1 \end{array}$$

w.r.t. the standard measure on $\mathbb{R}^n$.

$\Rightarrow \operatorname{vol}(X_1) = 2^s \cdot \operatorname{vol}(Y_1)$

(14) $\underline{\{\pm 1\}^r} \times \underline{[0,2\pi[^s} \times \underline{]0,1]} \times \underline{[0,1[^{r+s-1}} \xrightarrow{\sim} X_1 \xrightarrow{\sim} Y_1$

$((\varepsilon_j)_j, (\vartheta_j)_j, t, (\lambda_j)_j) \longmapsto \boxed{\times} \longmapsto \boxed{\times}$

$\longmapsto (\varepsilon_1, \dots, \varepsilon_r, e^{i\vartheta_1}, \dots, e^{i\vartheta_s}, e^{-i\vartheta_1}, \dots, e^{-i\vartheta_s}) \cdot t \cdot (E_{\sigma_1}, \dots, E_{\sigma_n})$

for $\underline{E_\sigma := \exp\left(\sum_{j=1}^{r+s-1} \lambda_j \, r_{j,\sigma}\right) \in K_1}$

$\mapsto \Big(\underline{\varepsilon_1} t E_{\sigma_1}, \dots, \underline{\varepsilon_r} t E_{\sigma_r}, \cos\vartheta_1 \cdot t \cdot E_{\sigma_{r+1}}, \sin\vartheta_1 \cdot t \cdot E_{\sigma_{r+1}}, \dots\dots, \sin\vartheta_s \cdot t \cdot E_{\sigma_{r+s}} \Big)$

(15) $\underline{\operatorname{vol}(X_1)} = 2^s \cdot \operatorname{vol}(Y_1) =$

$$= 2^s \cdot 2^r \cdot \int_{[0,2\pi[^s \times]0,1] \times [0,1[^{r+s-1}} 1 \cdot d(tE_{\sigma_1}) \cdots d(tE_{\sigma_r}) \cdot d(\underline{\cos\vartheta_1}\, t E_{\sigma_{r+1}}) \cdots d(\underline{\sin\vartheta_s} \cdot t \cdot E_{\sigma_{r+s}})$$

|

$$d(\cos\vartheta \cdot x)\cdot d(\sin\vartheta\cdot x) = x\cdot d\vartheta\cdot dx$$

$$= 2^{s}\cdot 2^{u}\cdot (2\pi)^{s}\cdot \int_{[0,1]^{r+s}} d(tE_{\sigma_1})\cdots d(tE_{\sigma_r})\cdot tE_{\sigma_{r+1}}\cdot d(tE_{\sigma_{r+1}})\cdots tE_{\sigma_{r+s}}\cdot d(tE_{\sigma_{r+s}})$$

(16)

$$\frac{\partial}{\partial t}(tE_{\sigma_i}) = E_{\sigma_i}$$

$$\frac{\partial}{\partial \lambda_j}(tE_{\sigma_i}) = tE_{\sigma_i}\cdot \gamma_{j,\sigma_i}$$

$$\Rightarrow |\det(\text{Jacobimatrix})| = \left|\det\begin{pmatrix} E_{\sigma_1} & tE_{\sigma_1}\gamma_{1,\sigma_1} & \cdots & tE_{\sigma_1}\cdot\gamma_{r+s-1,\sigma_1} \\ \vdots & \vdots & & \vdots \\ E_{\sigma_{r+s}} & tE_{\sigma_{r+s}}\gamma_{1,\sigma_{r+s}} & \cdots & tE_{\sigma_{r+s}}\cdot\gamma_{r+s-1,\sigma_{r+s}} \end{pmatrix}\right|$$

$$= E_{\sigma_1}\cdots E_{\sigma_{r+s}}\cdot t^{r+s-1}\cdot \left|\det\begin{pmatrix} 1 & \gamma_{1,\sigma_1} & \cdots & \gamma_{r+s-1,\sigma_1} \\ \vdots & \vdots & & \vdots \\ 1 & \gamma_{1,\sigma_{r+s}} & \cdots & \gamma_{r+s-1,\sigma_{r+s}} \end{pmatrix}\right|$$

$$= E_{\sigma_1}\cdots E_{\sigma_{r+s}}\cdot t^{r+s-1}\cdot 2^{-s}\cdot n\cdot R.$$

(17) $\Rightarrow \operatorname{vol}(X_1) = 2^r \cdot (2\pi)^s \cdot R \cdot \int_{[0,1]^{r+s}} \langle t^s E_{\sigma_{r+1}} \cdots E_{\sigma_{r+s}} \rangle \, E_{\sigma_1} \cdots E_{\sigma_{r+s}} \cdot t^{r+s-1} \cdot n \cdot dt \, d\lambda_1 \cdots d\lambda_{r+s-1}.$

$$= 2^r \cdot (2\pi)^s \cdot R \cdot \underbrace{\left(\int_0^1 t^{n-1} \cdot n \cdot dt \right)}_{t^n \big|_0^1 = 1} \cdot \int_{[0,1]^{r+s-1}} \underbrace{E_{\sigma_1}^{\rho_1} \cdots E_{\sigma_{r+s}}^{\rho_{r+s}}}_{\shortparallel} \, d\lambda_1 \cdots d\lambda_{r+s-1}$$

$$\exp\left(\sum_{i=1}^{r+s} \rho_i \left(\sum_{j=1}^{r+s-1} \lambda_j \gamma_{j,\sigma_i} \right) \right)$$

$$= \exp\left(\sum_{j=1}^{r+s-1} \lambda_j \underbrace{\left(\sum_{i=1}^{r+s} \rho_i \gamma_{j,\sigma_i} \right)}_{=0} \right)$$

because $\gamma_j \in H$.

(18) $\Rightarrow \boxed{\operatorname{vol}(X_1) = 2^r \cdot (2\pi)^s \cdot R}$

(19) $d_{\mathfrak n}(m) \overset{(r)}{=} \frac{1}{w}\cdot \#\left(j(\mathfrak n) \cap X_{\sqrt[n]{m}}\right) = \frac{1}{w}\cdot \#\left(j(\mathfrak n)\cap \sqrt[n]{m}\cdot X_1\right)$

$$\overset{(10)}{=} \frac{1}{w}\cdot\left(\frac{\mathrm{vol}(X_1)}{\mathrm{vol}(K_{\mathbb R}/j(\mathfrak n))}\cdot\left[\sqrt[n]{m}\right]^n + O\left(\sqrt[n]{m}^{\,n-1}\right)\right)$$

$$= \boxed{\frac{1}{w}\cdot\frac{2^r(2\pi)^s\cdot R}{\mathrm{Nm}(\mathfrak n)\cdot\sqrt{d_K}}}\cdot m + O\left(m^{1-\frac{1}{n}}\right),$$

$=: c_{\mathfrak n}$

(20) $\zeta_{\mathfrak n}(s) = \sum_{\substack{0\neq x\in\mathfrak n \\ \mathrm{mod}\ \mathcal O_K^\times}} |\mathrm{Nm}(x)|^{-s} = \sum_{m\ge 1} \underbrace{\#\{x\in\mathfrak n \bmod \mathcal O_K^\times \mid |\mathrm{Nm}(x)| = m\}}_{d_{\mathfrak n}(m)-d_{\mathfrak n}(m-1)}\cdot m^{-s}.$

$$= \sum_{m\ge1}\left(d_{\mathfrak n}(m) - d_{\mathfrak n}(m-1)\right) m^{-s}$$

$$= \sum_{m\ge1} d_{\mathfrak n}(m)\cdot\left(m^{-s}-(m+1)^{-s}\right)$$

$$= \sum_{m\ge1}\left[c_{\mathfrak n}\cdot m + O\left(m^{1-\frac1n}\right)\right]\cdot\int_m^{m+1} s\cdot x^{-s-1}\,dx$$

$$= \int_1^{\infty}\left(c_{\mathfrak n}\cdot\lfloor x\rfloor + O\left(x^{1-\frac1n}\right)\right)\cdot s\cdot x^{-s-1}\,dx$$

$m\le x< m+1 \Leftrightarrow m = \lfloor x\rfloor$

$$= \int_1^\infty \left(c_{\mathfrak{a}} \cdot x + \cancel{O(1)} + O\left(x^{1-\frac{1}{n}}\right) \right) s x^{-s-1}\, dx$$

$$= c_{\mathfrak{a}} \cdot \int_1^\infty s x^{-s}\, dx + O\left(\int_1^\infty s x^{-s-\frac{1}{n}}\, dx \right)$$

$$= c_{\mathfrak{a}} \cdot \left. \frac{s \cdot x^{1-s}}{1-s} \right|_1^\infty + O\left(\left. \frac{s \cdot x^{1-s-\frac{1}{n}}}{1-s-\frac{1}{n}} \right|_1^\infty \right)$$

$$= c_{\mathfrak{a}} \cdot \frac{s}{s-1} + O\left(\begin{array}{c} \text{uniformly absolutely convergent} \\ \text{for } \mathrm{Re}(s) \geq \sigma > 1-\frac{1}{n} \end{array} \right)$$

(21) So $\zeta_{\mathfrak{a}}(s) = c_{\mathfrak{a}} \cdot \frac{1}{s-1} + \left(\text{holomorphic on } \mathrm{Re}(s) > 1 - \frac{1}{n} \right)$

(22) $$\zeta_K(s) = \sum_{i=1}^{n} \left(Nm(\mathfrak{a}_i)\right)^s \cdot \zeta_{\mathfrak{a}_i}(s)$$

$$= \sum_{i=1}^{n} \left(Nm(\mathfrak{a}_i)\right)^s \cdot \frac{c_{\mathfrak{a}_i}}{s-1} + \text{holo for } \mathrm{Re}(s) > 1 - \frac{1}{n}.$$

$\Rightarrow$ meromorphic extension.

Z3 $\quad \operatorname{res}_{s=1}[J_K(s)] = \sum_{i=1}^{h} Nm(\mathfrak{a}_i)\cdot c_{\mathfrak{a}_i} = \sum_{i=1}^{h} Nm(\mathfrak{a}_i)\cdot \frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot R}{w\cdot Nm(\mathfrak{a}_i)\cdot\sqrt{|d_K|}}$

$$= \boxed{\frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot R\cdot h}{w\cdot\sqrt{|d_K|}}}$$

qed