Recall: $\sum_{\mathfrak{p}\in P} \mathrm{Nm}(\mathfrak{p})^{-s} = \log\left(\frac{1}{s-1}\right) + O(1)$

$= \log \zeta_K(s) + O(1)$

7.4 Dirichlet density

Consider a number field K and a subset A of the set P of maximal ideals of $\mathcal{O}_K$.

Definition 7.4.1: (a) The value

$$\overline{\mu}(A) := \limsup_{s\to 1+} \boxed{\frac{\sum_{\mathfrak{p}\in A} \mathrm{Nm}(\mathfrak{p})^{-s}}{\boxed{\sum_{\mathfrak{p}\in P} \mathrm{Nm}(\mathfrak{p})^{-s}}}}$$

$s \in \mathbb{R}$.

$\in [0,1]$

is called the *upper Dirichlet density of A*.

(b) The value

$$\underline{\mu}(A) := \liminf_{s\to 1+} \frac{\sum_{\mathfrak{p}\in A} \mathrm{Nm}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in P} \mathrm{Nm}(\mathfrak{p})^{-s}}$$

is called the *lower Dirichlet density of A.*

(c) If these coincide, their common value

$$\mu(A) := \lim_{s\to 1+} \frac{\sum_{\mathfrak{p}\in A} \mathrm{Nm}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in P} \mathrm{Nm}(\mathfrak{p})^{-s}} = \lim_{s\to 1+} \frac{\sum_{\mathfrak{p}\in A} \mathrm{Nm}(\mathfrak{p})^{-s}}{\log\left(\frac{1}{s-1}\right)} \Rightarrow (c)$$

is called the *Dirichlet density of A.*

Proposition 7.4.2: (a) We have $0 \leqslant \underline{\mu}(A) \leqslant \overline{\mu}(A) \leqslant 1$.

(b) For any subset $B \subset A$ we have $\overline{\mu}(B) \leqslant \overline{\mu}(A)$ and $\underline{\mu}(B) \leqslant \underline{\mu}(A)$, and also $\mu(B) \leqslant \mu(A)$ if these exist.

(c) We have $\mu(A) = 0$ if A is finite.

(d) We have $\mu(A) = 1$ if $P \smallsetminus A$ is finite.

(e) For any disjoint subsets $A, B \subset P$, if two of $\mu(A)$, $\mu(B)$, $\mu(A \cup B)$ exist, then so does the third and we have $\mu(A) + \mu(B) = \mu(A \cup B)$.

See above.

$$\sum_{\mathfrak{p} \in A \cup B} = \sum_{\mathfrak{p} \in A} + \sum_{\mathfrak{p} \in B}.$$

$$\mu(P) = \lim_{s \to 1+} 1 = 1.$$

(e) $\parallel$

$\mu(A) + \underbrace{\mu(P \smallsetminus A)}_{\text{zero if } P \smallsetminus A \text{ finite.}}$

qed.

Proposition-Definition 7.4.3: If the *natural density of A*

$$\gamma(A) \;:=\; \lim_{x\to\infty} \frac{\overbrace{\left|\{\mathfrak{p}\in A \mid \mathrm{Nm}(\mathfrak{p})\leqslant x\}\right|}^{=:\,a(x)}}{\underbrace{\left|\{\mathfrak{p}\in P \mid \mathrm{Nm}(\mathfrak{p})\leqslant x\}\right|}_{=:\,b(x)}}$$

$\Rightarrow$ $a(x) = (\gamma(A)+o(1))\cdot b(x)$

for $x\to\infty$

exists, so does the Dirichlet density $\mu(A)$ and they are equal.

Proof:

$$\sum_{\mathfrak{p}\in A}(\mathrm{Nm}\,\mathfrak{p})^{-s} = \sum_{m\geq 1}\#\{\mathfrak{p}\in A \mid \mathrm{Nm}(\mathfrak{p})=m\}\cdot m^{-s}$$

$$= \sum_{m\geq 1}\big(a(m)-a(m-1)\big)\cdot m^{-s}$$

$$= \sum_{m\geq 1} a(m)\cdot\big(m^{-s}-(m+1)^{-s}\big) - \underset{\overset{\shortparallel}{0}}{a(0)}\cdot 1$$

$$= \sum_{m\geq 1} a(m)\cdot\int_m^{m+1} s\cdot x^{-s-1}\,dx$$

(*)

$$= \int_1^\infty a(x)\cdot s\cdot x^{-s-1}\,dx$$

$$= \int_1^\infty (\gamma(A)+o(1))\cdot b(x)\cdot s\,x^{-s-1}\,dx$$

$$= \gamma(A)\cdot\underbrace{\int_1^\infty b(x)\,s\,x^{-s-1}\,dx}_{\overset{(*)}{=}\sum_{\mathfrak{p}\in P}\mathrm{Nm}(\mathfrak{p})^{-s}} + \underbrace{\int_1^\infty \underbrace{o(1)\cdot b(x)}_{b(x)}\,s\,x^{-s-1}\,dx}$$

(**)

For any $\varepsilon > 0$ take $x_0 \geq 1$ such that $\forall x \geq x_0$: this has absolute value $\leq \varepsilon$.

$$\Rightarrow \left| (**) \right| \leq \int_1^{x_0} \underset{\text{constant.}}{\overset{\shortparallel}{C}} \cdot s\, x^{-s-1}\, dx + \int_{x_0}^{\infty} \varepsilon \cdot s \cdot x^{-s-1}\, dx$$

$$= O(1) + \varepsilon \cdot \left(\log\left(\frac{1}{s-1}\right) + O(1) \right)$$

$$\Rightarrow \left| \frac{(**)}{\sum_{\mathfrak{p} \in P} N_m(\mathfrak{p})^{-s}} \right| \leq \frac{\varepsilon \cdot \log\left(\frac{1}{s-1}\right) + O(1)}{\log\left(\frac{1}{s-1}\right) + O(1)} = \varepsilon + o(1).$$

$$\limsup_{s \to 1+} \left| \frac{\sum_{\mathfrak{p} \in A} N_m(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p} \in P} N_m(\mathfrak{p})^{-s}} - \delta(A) \right| \leq \limsup_{s \to 1+} (\varepsilon + o(1)) = \varepsilon.$$

$$\text{Vary } \varepsilon \Rightarrow \limsup_{s \to 1+} \left| \ \dots \ \right| = 0.$$

$\underline{\text{qed}}$.