

K/\mathbb{Q} Galois with group $(\mathbb{Z}/N\mathbb{Z})^\times$

Theorem 7.6.9: The zeta function $\zeta_K(s)$ of the field $K := \mathbb{Q}(\mu_N)$ is the product of the L -functions $L(\chi, s)$ for all primitive Dirichlet characters χ of conductor dividing N .

Proof: $\zeta_K(s) = \prod_{\mathfrak{f} \subset \mathcal{O}_K} (1 - N_{\mathfrak{f}}(\mathfrak{p})^{-s})^{-1} = \prod_{\mathfrak{p}} \left(\prod_{\mathfrak{f}|\mathfrak{p}} (1 - N_{\mathfrak{f}}(\mathfrak{p})^{-s})^{-1} \right)$

$(1 - (p^{f_p})^{-s})^{-v_p}$

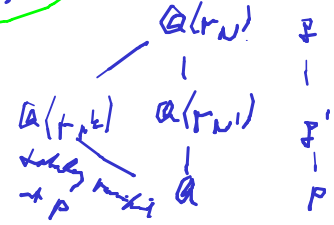
$\varphi(N) = v_p e_p f_p$
 $e_p = 1 \Leftrightarrow p \nmid N$
 and then
 $f_p = \text{order of } [p] \text{ in } (\mathbb{Z}/N\mathbb{Z})^\times$.

$p \nmid N \Rightarrow \prod_{\chi} (1 - \chi(p) p^{-s}) = \prod_{\mathfrak{f} \in M_{f_p}} (1 - \mathfrak{f} p^{-s})^{-v_p}$

also $f_p = [p]$
 $(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\pi} \mathbb{C}^\times$

$\mathfrak{f} \in M_{f_p}$
 each occurs v_p times

$p \mid N \Rightarrow$ same for N' instead of N
 $N = p^k \cdot N'$ with $p \nmid N'$.



$\mathbb{Z} \rightarrow \mathbb{Z}' \rightarrow \mathbb{Z}$ } totally ramified.

$e_p = 1 \Rightarrow \varphi(N) = v_p \cdot f_p$

good

Theorem 7.6.10: For any non-principal Dirichlet character χ we have $L(\chi, 1) \neq 0$.

Proof:

$$\zeta_Q(s) = \zeta(s) \cdot \prod_{\chi \neq 1} L(\chi, s)$$

(7.6.5)
 $\chi \neq 1$

simple pole at $s=1$
holomorphic at $s=1$

\Rightarrow no zero.

qed.

Proposition 7.6.11: For any non-principal Dirichlet character χ we have

$$\sum_{p \text{ prime}} \chi(p) p^{-s} = O(1) \text{ for real } s \rightarrow 1+.$$

Proof: $\log L(\chi, s) = - \sum_p \sum_{k \geq 1} \frac{(\chi(p) p^{-s})^k}{k}$

\Downarrow
 $O(1)$ near $s=1$

\Downarrow
 because $L(\chi, 1) \neq 0$

$$= \sum_p \chi(p) p^{-s} + \sum_p \sum_{k \geq 2} O(p^{-sk}) = O(1) \sim \Re(s) > \frac{2}{3}$$

qed.

7.7 Primes in arithmetic progressions

Theorem 7.7.1: For any coprime integers a and $N \geq 1$ the set of rational primes $p \equiv a \pmod{N}$ has Dirichlet density $\frac{1}{\varphi(N)}$. In particular it is infinite.

↙ S_a

$$\sum_{\chi \text{ of modulus } N} \chi(a)^{-1} \chi(p) = \sum_{\chi} \chi(g) = \begin{cases} \varphi(N) & \text{if } g = [1] \\ & \Leftrightarrow [p] = [a] \\ 0 & \text{else} \end{cases}$$

$\underline{p}_{\text{mod } N}: p \not\equiv 1 \pmod{N} \rightarrow$
 $[a], [p] \in (\mathbb{Z}/N\mathbb{Z})^\times$
 $g := \frac{[p]}{[a]}$

$(\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$
 $\downarrow \chi$
 $g \mapsto \sum_{j \in \mathbb{Z}} \chi^j, \chi = \omega(g)$

$\sum_{j \in \mathbb{Z}} \chi^j = \begin{cases} \varphi(N) & \text{if } \chi = 1 \\ 0 & \text{else} \end{cases}$
 $\chi^{p-1} + \chi^{p-2} + \dots + \chi + 1$

$$\sum_{p \in S_a} p^{-s} = \sum_{p \not\equiv 1 \pmod{N}} \frac{1}{\varphi(N)} \cdot \sum_{\chi} \chi(a)^{-1} \chi(p) \cdot p^{-s} =$$

$$= \frac{1}{\varphi(N)} \cdot \sum_{\chi} \chi(a)^{-1} \cdot \sum_{p \not\equiv 1 \pmod{N}} \chi(p) p^{-s} = \begin{cases} 0(1) & \text{if } \chi \neq 1 \\ \log\left(\frac{1}{s-1}\right) + 0(1) & \text{if } \chi = 1 \end{cases}$$

$$= \frac{1}{\varphi(N)} \cdot \log\left(\frac{1}{s-1}\right) + 0(1)$$

$$\sum_p p^{-s} = \log\left(\frac{1}{s-1}\right) + 0(1) \Rightarrow \frac{\sum_{p \in S_a} p^{-s}}{\sum_p p^{-s}} = \frac{1}{\varphi(N)} + o(1) \text{ as } s \rightarrow \infty$$

$\Rightarrow \mu(S_a) = \frac{1}{\varphi(N)}$ qed.