$K/\mathbb{Q}$ galois with group $(\mathbb{Z}/N\mathbb{Z})^\times$

Theorem 7.6.9: The zeta function $\zeta_K(s)$ of the field $K := \mathbb{Q}(\mu_N)$ is the product of the L-functions $L(\chi, s)$ for all primitive Dirichlet characters χ of conductor dividing N.

$\varphi(N) = r_p e_p f_p$

Proof:
$$\zeta_K(s) = \prod_{\mathfrak{p}\subset\mathcal{O}_K} \left(1 - N_m(\mathfrak{p})^{-s}\right)^{-1} = \prod_p \left(\underbrace{\prod_{\mathfrak{p}|p} \left(1-N_m(\mathfrak{p})^{-s}\right)^{-1}}_{\left(1-(p^{f_p})^{-s}\right)^{-r_p}}\right).$$

$e_p = 1 \Leftrightarrow p\nmid N$ and then f_p = order of $[p]$ in $(\mathbb{Z}/N\mathbb{Z})^\times$.

$$p\nmid N \Rightarrow \underbrace{\prod_\chi \left(1-\chi(p)p^{-s}\right)}_{} = \underbrace{\prod_{\zeta\in\mu_{f_p}} \left(1-\zeta p^{-s}\right)^{r_p}}_{} \;//\; \left(1-(p^{f_p})^{-s}\right)^{-r_p}$$

$$\begin{tikzcd} \text{order } f_p \quad [p] \arrow[r, mapsto] & \zeta\in\mu_{f_p} \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\chi"'] & \mathbb{C}^\times \end{tikzcd}$$

each occurs r_p times

$e_p = 1 \Rightarrow \varphi(N) = r_p\cdot f_p$

$p\,|\,N \Rightarrow$ same for N' instead of N
$N = p^k\cdot N'$ with $p\nmid N'$.

$$\begin{tikzcd} & \mathbb{Q}(\mu_N) \arrow[dl, dash] \arrow[d, dash] & \mathfrak{p} \arrow[d, dash] \\ \mathbb{Q}(\mu_{p^k}) \arrow[dr, dash] & \mathbb{Q}(\mu_{N'}) \arrow[d, dash] & \mathfrak{p}' \arrow[d, dash] \\ & \mathbb{Q} & p \end{tikzcd}$$

$\Big\}$ totally ramified.

$\mathbb{Q}(\mu_{p^k})$ totally ramified at p

qed.

Theorem 7.6.10: For any non-principal Dirichlet character χ we have $L(\chi, 1) \neq 0$.

Proof:

$$\underbrace{\zeta_{\mathbb{Q}(\mu_N)}(s)}_{\text{simple pole at } s=1} \overset{(7.6.5)}{=} \underbrace{\zeta(s)} \cdot \prod_{\chi \neq 1} \underbrace{L(\chi, s)}_{\text{holomorphic at } s=1}$$

$\Rightarrow$ no zero.

qed.

Proposition 7.6.11: For any non-principal Dirichlet character χ we have

$$\boxed{\sum_{p \text{ prime}} \chi(p)p^{-s}} = O(1) \quad \text{for real } s \to 1+.$$

Proof:

$$\log L(\chi, s) = -\sum_p \log(1-\chi(p)p^{-s}) = \sum_p \sum_{k\geq 1} \frac{(\chi(p)p^{-s})^k}{k}$$

$\log L(\chi,s) = O(1)$ near $s=1$ since $L(\chi,1) \neq 0$

$$= \boxed{\sum_p \chi(p) p^{-s}} + \underbrace{\sum_p \sum_{k \geq 2} O(p^{-sk})}_{} = O(1) \leftarrow \mathrm{Re}(s) > \tfrac{2}{3}.$$

qed.

7.7 Primes in arithmetic progressions

S_a

Theorem 7.7.1: For any coprime integers a and $N \geqslant 1$ the set of rational primes $p \equiv a \bmod (N)$ has Dirichlet density $\frac{1}{\varphi(N)}$. In particular it is infinite.

Pf: $p \nmid N \rightsquigarrow \boxed{\sum_{\chi \text{ of modulus } N} \chi(a)^{-1} \cdot \chi(p)} = \sum_{\chi} \chi(g) = \boxed{\begin{cases} \varphi(N) & \text{if } g = [1]. \iff [p] = [a] \\ \text{else: } \sum_{\gamma \in \mu_f} \gamma \cdot \frac{\varphi(N)}{f} = 0 \end{cases}}$

$[a], [p] \in (\mathbb{Z}/N\mathbb{Z})^{\times}$

$g := \frac{[p]}{[a]}$

$$(\mathbb{Z}/N\mathbb{Z})^{\times} \longrightarrow \mathbb{C}^{\times}$$

$$g \longmapsto \gamma \in \mu_f, \quad f = \mathrm{ord}(g)$$

γ root of $X^{f-1} + X^{f-2} + \dots + X + 1.$

$$\sum_{p \in S_a} p^{-s} = \sum_{p \nmid N} \frac{1}{\varphi(N)} \cdot \sum_{\chi} \chi(a)^{-1} \chi(p) \cdot p^{-s} =$$

$$= \frac{1}{\varphi(N)} \cdot \sum_{\chi} \underbrace{\chi(a)^{-1}}\cdot \underbrace{\sum_{p \nmid N} \chi(p) p^{-s}} = \begin{cases} O(1) & \text{if } \chi \neq 1 \\ \log\left(\frac{1}{s-1}\right) + O(1) & \text{if } \chi = 1 \end{cases}$$

$$= \frac{1}{\varphi(N)} \cdot \log\left(\frac{1}{s-1}\right) + O(1)$$

$$\sum_{p} p^{-s} = \log\left(\frac{1}{s-1}\right) + O(1)$$

$$\Rightarrow \frac{\sum_{p \in S_a} p^{-s}}{\sum_{p} p^{-s}} = \frac{1}{\varphi(N)} + o(1) \text{ for } s \to 1$$

$$\Rightarrow \mu(S_a) = \frac{1}{\varphi(N)}. \quad \text{qed.}$$