Recall:

Theorem 7.7.1: For any coprime integers a and $N \geqslant 1$ the set of rational primes $p \equiv a \bmod (N)$ has Dirichlet density $\frac{1}{\varphi(N)}$. In particular it is infinite.

$\rightsquigarrow$ Behaviour of p in $\mathbb{Q}(\mu_N)$.

$\mathrm{Frob}_p \in [p] \in (\mathbb{Z}/N\mathbb{Z})^\times \overset{\sim}{=} \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$

7.8 Bonus Material: Abelian Artin L-functions

Consider an abelian extension of number fields L/K with Galois group Γ. Then for any prime $\mathfrak{q}$ of $\mathcal{O}_L$, the decomposition group $\Gamma_{\mathfrak{q}}$, the inertia group $I_{\mathfrak{q}}$, and the Frobenius substitution $\mathrm{Frob}_{\mathfrak{q}}$ depend only on the underlying prime $\mathfrak{p}$ of $\mathcal{O}_K$. We therefore denote them also by $\Gamma_{\mathfrak{p}}$, $I_{\mathfrak{p}}$, $\mathrm{Frob}_{\mathfrak{p}}$ respectively.

Definition 7.8.1: The *Artin L-function* associated to any homomorphism $\chi\colon \Gamma \to \mathbb{C}^\times$ is

$$\boxed{L_K(\chi, s) \;:=\; \prod_{\substack{\mathfrak{p} \\ \chi|I_{\mathfrak{p}}=1}} \big(1 - \chi(\mathrm{Frob}_{\mathfrak{p}})\,\mathrm{Nm}(\mathfrak{p})^{-s}\big)^{-1}.}$$

Example 7.8.2: In the case $K = \mathbb{Q}$ and $L = \mathbb{Q}(\mu_N)$ and the usual identification $\Gamma \cong (\mathbb{Z}/N\mathbb{Z})^\times$ the Artin L-function $L_K(\chi, s)$ is the Dirichlet L-function $L(\chi, s)$ for the primitive Dirichlet character associated to χ.

Proposition 7.8.3: This product converges absolutely and locally uniformly for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a holomorphic function there.

because $|\chi(\mathrm{Frob}_{\mathfrak{p}})| = 1$.

Proposition 7.8.4: For the trivial homomorphism χ we have $L_K(\chi, s) = \zeta_K(s)$.

Proposition 7.8.5: The zeta function $\zeta_L(s)$ is the product of the L-functions $L_K(\chi, s)$ for all χ.

Recall: $\forall\, \mathfrak{p} \subset \mathcal{O}_K$: $\displaystyle\prod_{\mathfrak{q}\mid\mathfrak{p}} \left(1 - \mathrm{Nm}(\mathfrak{q})^{-s}\right) = \left(1 - \mathrm{Nm}(\mathfrak{p})^{-fs}\right)^r = \prod_{\zeta \in \mu_f} \left(1 - \zeta \cdot \mathrm{Nm}(\mathfrak{p})^{-s}\right)^r$

$n = |\Gamma| = ref$

qed

Theorem 7.8.6: For every non-trivial χ the function $L_K(\chi, s)$ extends uniquely to a holomorphic function on the region $\mathrm{Re}(s) > 1 - \frac{1}{[K/\mathbb{Q}]}$.

(without proof)

Theorem 7.8.7: For every non-trivial χ we have $L_K(\chi, 1) \neq 0$.

Proof: $\displaystyle\underbrace{\zeta_L(s)}_{\text{simple pole}} = \underbrace{\zeta_K(s)}_{\text{simple pole}} \cdot \prod_{\chi \neq 1} \underbrace{L_K(\chi, s)}_{\text{holomorphic.}}$

qed.

7.9 Bonus Material: Cebotarev density theorem

Consider an arbitrary Galois extension of number fields L/K with Galois group Γ. For any $\gamma \in \Gamma$ we denote the conjugacy class by $O_\Gamma(\gamma) := \{{}^\delta\gamma \mid \delta \in \Gamma\}$ and let P_γ denote the set of primes $\mathfrak{p} \subset \mathcal{O}_K$ that are unramified in $\mathcal{O}_L$ and whose Frobenius substitution for some (and equivalently every) $\mathfrak{q}|\mathfrak{p}$ lies in $O_\Gamma(\gamma)$.

Theorem 7.9.1: The set $\boxed{P_\gamma \text{ has the Dirichlet density } \frac{|O_\Gamma(\gamma)|}{|\Gamma|}.}$

Steps in the proof of Theorem 7.9.1: Let $C(L/K, \gamma)$ denote the statement of 7.9.1.

$$\begin{array}{ll} L & \mathfrak{q} \\ \big| \; \Big\} \text{ cyclic with group } \langle\gamma\rangle & \\ L^{\langle\gamma\rangle} =: K' & \gamma \\ \big| & \\ K & \mathfrak{p} \end{array}$$

(*) • If $C(L/L^{\langle\gamma\rangle}, \gamma)$ holds, then so does $C(L/K, \gamma)$.

Proof: Let $Q_\gamma := \{\mathfrak{q} \subset \mathcal{O}_L \mid \text{unramified over } \mathfrak{p} := \mathfrak{q} \cap \mathcal{O}_K,\ \mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}} = \gamma\}$

$\Rightarrow P_\gamma := \{\mathfrak{q} \cap \mathcal{O}_K \mid \mathfrak{q} \in Q_\gamma\}$.

Take $\mathfrak{p} \in P_\gamma$ and $\mathfrak{q} \in Q_\gamma$ above $\mathfrak{p}$:

Then $\forall \delta \in \Gamma$:
- ${}^\delta\mathfrak{q}$ lies above $\mathfrak{p}$
- " unramified over $\mathfrak{p}$
- $\mathrm{Frob}_{{}^\delta\mathfrak{q}|\mathfrak{p}} = {}^\delta(\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}}) = {}^\delta\gamma = \gamma$ iff $\delta \in \mathrm{Cent}_\Gamma(\gamma)$
- ${}^\delta\mathfrak{q} = \mathfrak{q}$ iff $\delta \in \Gamma_\mathfrak{q} = \langle\gamma\rangle$.

$\Rightarrow \forall \mathfrak{p} \in P_\gamma : \#\{\mathfrak{q} \in Q_\gamma \mid \mathfrak{q} \cap \mathcal{O}_K = \mathfrak{p}\} = [\mathrm{Cent}_\Gamma(\gamma) : \langle\gamma\rangle]$.

Let $Q'_f := \{ \mathfrak{q} \subset \mathcal{O}_L \mid$ unramified over $\mathfrak{p}' := \mathfrak{q} \cap \mathcal{O}_{K'}$, $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}'} = f \}$

$P'_f := \{ \mathfrak{q} \cap \mathcal{O}_{K'} \mid \mathfrak{q} \in Q'_f \}$.

Then $\forall \mathfrak{q} \in Q_f : \mathfrak{q} \in Q'_f$.

$\forall \mathfrak{q} \in Q'_f \setminus Q_f$: $e_{\mathfrak{p}'|\mathfrak{p}} > 1$ or $f_{\mathfrak{p}'|\mathfrak{p}} > 1$. } Contribute 0 to Dirichlet density on K'.

$$\frac{\sum_{\mathfrak{p} \in P_f} \mathrm{Nm}(\mathfrak{p})^{-s}}{\log\left(\frac{1}{s-1}\right)} = \frac{1}{[\mathrm{Cent}_\Gamma(f) : \langle f \rangle]} \cdot \frac{\sum_{\mathfrak{q} \in Q_f} \mathrm{Nm}(\mathfrak{q} \cap \mathcal{O}_{K'})^{-s}}{\log\left(\frac{1}{s-1}\right)}$$

$$= \frac{[\Gamma : \mathrm{Cent}_\Gamma(f)]}{[\Gamma : \langle f \rangle]} \cdot \frac{\sum_{\mathfrak{q} \in Q'_f} \mathrm{Nm}(\mathfrak{q} \cap \mathcal{O}_{K'})^{-s} + O(1)}{\log\left(\frac{1}{s-1}\right)}$$

$$= \frac{|\mathcal{O}_\Gamma(f)|}{|\Gamma|} \cdot |\langle f \rangle| \cdot \Big(\underbrace{\mu(P'_f)}_{= \frac{1}{|\langle f \rangle|}} + o(1)\Big)$$

$$= \frac{|\mathcal{O}_\Gamma(f)|}{|\Gamma|} + o(1) \qquad \Rightarrow \mu(P_f) = \frac{|\mathcal{O}_\Gamma(f)|}{|\Gamma|}$$

qed

- So for the rest we may assume that Γ is abelian; or even cyclic with $\Gamma = \langle \gamma \rangle$.
- For Γ abelian $C(L/K, \gamma)$ follows from Theorem 7.8.7.

- But Theorem 7.8.7 depends on Theorem 7.8.6, which we did not prove.

- Now consider the special case $L = K(\mu_m)$ for some $m > 1$.
  - Let J_m denote the group of fractional ideals of $\mathcal{O}_K$ that are coprime to m.

$$J_m = \{ \mathfrak{a}\mathfrak{b}^{-1} \mid 0 \neq \mathfrak{a},\mathfrak{b} < \mathcal{O}_K ;\ \mathfrak{a} + m\mathcal{O}_K = \mathfrak{b} + m\mathcal{O}_K = \mathcal{O}_K \}.$$

  - Then every ideal class can be represented by an element of J_m.

With $\mathfrak{a} = \prod_{i=1}^{s} \mathfrak{p}_i^{\nu_i}$ distinct $\mathfrak{p}_i$, $\nu_i \in \mathbb{Z}$ such that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ for $r \le s$ are the prime divisors of m.

$\forall i$: choose $\gamma_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j \setminus \mathfrak{p}_i^2$ by 6.3.1.

Let $\gamma := \prod_{i=1}^{r} \gamma_i^{\nu_i} \Rightarrow \gamma \cdot \mathcal{O}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{\nu_i} \cdot$ (product of other primes) $\Rightarrow \mathfrak{a} \sim \underbrace{\gamma^{-1}\mathfrak{a}}_{\in J_m} = \prod_{i=r+1}^{s} \mathfrak{p}_i^{\nu_i} \cdot$ primes $\neq \mathfrak{p}_1, \dots, \mathfrak{p}_r$.

  - Let P_m denote the subgroup of all principal ideals $(\frac{x}{y})$ for $x, y \in 1 + m\mathcal{O}_K$.
  - Then J_m/P_m is finite.

$\mathrm{Cl}(\mathcal{O}_K)$ finite, so enough: J_m^1/P_m finite for $J_m^1 := \{\mathfrak{a} \in J_m \text{ principal}\} \ni \mathfrak{a}$.

Then $\mathfrak{a} = \prod \mathfrak{p}_i^{\nu_i}$ as above $\Rightarrow \forall i \le r: \nu_i = 0$

Let $\eta := \prod \mathfrak{p}_i^{\max\{0,-\nu_i\}} \in \mathcal{O}_K$, coprime to m; $\eta\mathfrak{a} < \mathcal{O}_K$ coprime to m, $\eta\mathfrak{a} = \langle \gamma \rangle$ for $\gamma \in \mathcal{O}_K$

$\Rightarrow \mathfrak{a} = \langle \frac{\gamma}{\eta} \rangle$ with $\gamma, \eta \in \mathcal{O}_K$ coprime to m.

$(\mathcal{O}_K/m\mathcal{O}_K)^\times$ finite.

  - We identify $\Gamma = \mathrm{Gal}(L/K)$ with a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ as usual.
  - There is a well-defined homomorphism

$$\boxed{\bar{N} : J_m/P_m \to \Gamma < (\mathbb{Z}/m\mathbb{Z})^\times, \quad [\mathfrak{a}] \mapsto [\mathrm{Nm}(\mathfrak{a})].}$$

$\tau \mapsto [e_\tau]$ with $\tau\zeta = \zeta^{e_\tau}$ for any $\zeta \in \mu_m$.

Take $\mathfrak{p} < \mathcal{O}_K$ maximal coprime to m.

$\Rightarrow \mathrm{Nm}(\mathfrak{p})$ coprime to m.

$\Rightarrow \mathfrak{p} \nmid \mathrm{disc}(X^m - 1)$.

$\Rightarrow \mathfrak{p} \nmid \mathrm{disc}\, L/K \Rightarrow \mathfrak{p}$ unramified in L.

Take $\mathfrak{q} < \mathcal{O}_L$ above $\mathfrak{p}$ $\Rightarrow$ $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}} \longmapsto [\mathrm{Nm}(\mathfrak{q})] \in (\mathbb{Z}/m\mathbb{Z})^\times$.

$$\Rightarrow [\mathrm{Nm}(\mathfrak{q})] \in \Gamma.$$

Then $\mathfrak{p}$ generates J_m $\Rightarrow$ have: $J_m \longrightarrow \Gamma < (\mathbb{Z}/m\mathbb{Z})^\times$

$$\mathfrak{n} \longmapsto [\mathrm{Nm}(\mathfrak{n})].$$

This is trivial on P_m.

– As in the proof of the analytic class number formula one shows that, for some explicit positive real constant c that is independent of χ, we have for any homo $\chi : \Gamma \to \mathbb{C}^\times$.

$$L_K(\chi, s) = \begin{cases} 1 & \text{if } \chi \circ \bar{N} = 1 \\ 0 & \text{if } \chi \circ \bar{N} \neq 1 \end{cases} \cdot \frac{c}{s-1} + O(1) \qquad \text{for } \mathrm{Re}(s) > 1 - \frac{1}{[K/\mathbb{Q}]}.$$

$$(1+O(s-1)) \cdot L_K(\chi,s) = \prod_{\substack{\mathfrak{p} < \mathcal{O}_K \\ \mathfrak{p} \nmid \mathfrak{m}}} \left(1 - \chi(\mathrm{Nm}(\mathfrak{p})) \cdot \mathrm{Nm}(\mathfrak{p})^{-s}\right)^{-1}$$

for $\mathrm{Re}(s) > 0$

Choose rep's $\mathfrak{n}_i \in I_\mathfrak{m}$ for each ideal class in $Cl(\mathcal{O}_K)$.

$$= \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_K \\ \mathfrak{n}}} \chi(\mathrm{Nm}(\mathfrak{n})) \cdot \mathrm{Nm}(\mathfrak{n})^{-s}$$

$$= \sum_{i=1}^{h} \sum_{\substack{x \in \mathfrak{n}_i \\ \mathfrak{n}_i^{-1} x \text{ coprime to } \mathfrak{m} \\ x \bmod \mathcal{O}_K^\times}} \chi(\mathrm{Nm}(\mathfrak{n}_i^{-1} x)) \cdot \mathrm{Nm}(\mathfrak{n}_i^{-1} x)^{-s}$$

$$= \sum_{i=1}^{h} \mathrm{Nm}(\mathfrak{n}_i)^{s} \cdot \sum_{\dots} \chi(\mathrm{Nm}(\mathfrak{n}_i^{-1} x)) \cdot |\mathrm{Nm}_{K/\mathbb{Q}}(x)|^{-s}.$$

Set $U_\mathfrak{m} := \ker\left(\mathcal{O}_K^\times \to (\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)^\times\right)$ and $d := [\mathcal{O}_K^\times : U_\mathfrak{m}]$.

$$= \frac{1}{d} \sum_{i=1}^{h} \mathrm{Nm}(\mathfrak{n}_i)^{s} \cdot \sum_{\substack{x \in \mathfrak{n}_i \\ \mathfrak{n}_i^{-1} x \text{ coprime to } \mathfrak{m} \\ \bmod U_\mathfrak{m}}} \underbrace{\chi(\mathrm{Nm}(\mathfrak{n}_i^{-1} x))}_{\text{depends only on } x \bmod \mathfrak{n}_i \mathfrak{m}} \cdot |\mathrm{Nm}_{K/\mathbb{Q}}(x)|^{-s}$$

$$= \frac{1}{d} \cdot \sum_{i=1}^{h} Nm(n_i)^{s} \cdot \sum_{\substack{[y] \in n_i \text{ and } n_i m \\ n_i^{-1} y \text{ coprime to } m}} \chi\left(Nm(n_i^{-1} y)\right) \cdot \underbrace{\sum_{\substack{x \in y + n_i m \\ \text{mod } U_m}} |Nm_{K/\mathbb{Q}}(x)|^{-s}}_{\overbrace{\frac{c\,d}{Nm(n_i)} \cdot \left[\frac{1}{s-1} + O(1)\right]}^{\S 7.1}}$$

$$= \sum_{i=1}^{h} \underbrace{Nm(n_i)^{s-1}}_{\overset{\shortparallel}{1 + O(s-1)}} \cdot \underbrace{\left[\sum_{i=1}^{h} \sum_{[y]} \chi\left(Nm(n_i^{-1} y)\right)\right]}_{\underbrace{\sum_{[n] \in I_m/P_m} \chi\left([Nm(n)]\right)}_{= \begin{cases} [I_m : P_m] & \text{if } \chi \circ \bar{N} = 1 \\ 0 & \text{else.} \end{cases}}} \cdot \left(\frac{c}{s-1} + O(1)\right)$$