Number Theory I und II

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This summary contains the definitions and results covered in the lecture course, but no proofs, examples, explanations, or exercises.

Content

1 Some commutative algebra

1.1 Integral ring extensions

All rings are assumed to be commutative and unitary. Consider a ring extension $A \subset B$.

Definition 1.1.1: (a) An element $b \in B$ is called *integral over* A if there exists a monic $f \in A[X]$ with $f(b) = 0$.

(b) The ring B is called *integral over* A if every $b \in B$ is integral over A.

(c) The *integral closure of* A *in* B is the set $\tilde{A} := \{b \in B \mid b \text{ integral over } A\}$.

Definition-Example 1.1.2: (a) An element $z \in \mathbb{C}$ is integral over $\mathbb{Q}$ if and only if z is an *algebraic number*.

(b) An element $z \in \mathbb{C}$ is integral over $\mathbb{Z}$ if and only if z is an *algebraic integer*.

Proposition 1.1.3: The following statements for an element $b \in B$ are equivalent:

(a) b is integral over A.

(b) The subring $A[b] \subset B$ is finitely generated as an A-module.

(c) b is contained in a subring of B which is finitely generated as an A-module.

Proposition 1.1.4: (a) For any integral ring extensions $A \subset B$ and $B \subset C$ the ring extension $A \subset C$ is integral.

(b) The subset $\tilde{A}$ is a subring of B that contains A.

(c) The subring $\tilde{A}$ is its own integral closure in B.

1.2 Prime ideals

Consider an integral ring extension $A \subset B$.

Proposition 1.2.1: For every prime ideal $\mathfrak{q} \subset B$ the intersection $\mathfrak{q} \cap A$ is a prime ideal of A.

Definition 1.2.2: We say that $\mathfrak{q}$ *lies over* $\mathfrak{q} \cap A$.

Theorem 1.2.3: For any prime ideals $\mathfrak{q} \subset \mathfrak{q}' \subset B$ over the same $\mathfrak{p}$ we have $\mathfrak{q} = \mathfrak{q}'$.

Theorem 1.2.4: For every prime ideal $\mathfrak{p} \subset A$ there exists a prime ideal $\mathfrak{q} \subset B$ over $\mathfrak{p}$.

1.3 Normalization

From now on we assume that A is an integral domain with quotient field K.

Definition 1.3.1: (a) The integral closure of A in K is called the *normalization of* A.

(b) The ring A is called *normal* if this normalization is A.

Proposition 1.3.2: (a) The normalization of A is normal.

(b) Any unique factorization domain is normal.

1.4 Localization

Definition 1.4.1: A subset $S \subset A \setminus \{0\}$ is called *multiplicative* if it contains 1 and is closed under multiplication.

Definition-Proposition 1.4.2: For any multiplicative subset $S \subset A$ the subset

$$S^{-1}A \;:=\; \left\{ \tfrac{a}{s} \;\middle|\; a \in A,\; s \in S \right\}$$

is a subring of K that contains A and is called the *localization of A with respect to S*.

Example 1.4.3: For every prime ideal $\mathfrak{p} \subset A$ the subset $A \setminus \mathfrak{p}$ is multiplicative. The ring $A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1}A$ is called the *localization of A at $\mathfrak{p}$*.

Proposition 1.4.4: For every multiplicative subset $S \subset A$ we have:

(a) $S^{-1}\tilde{A} = \widetilde{S^{-1}A}$.

(b) If A is normal, then so is $S^{-1}A$.

1.5 Field extensions

In the following we consider a normal integral domain A with quotient field K, and an algebraic field extension L/K, and let B be the integral closure of A in L.

Proposition 1.5.1: For any homomorphism $\sigma\colon L \to M$ of field extensions of K, an element $x \in L$ is integral over A if and only if $\sigma(x)$ is integral over A.

Proposition 1.5.2: An element $x \in L$ is integral over A if and only if the minimal polynomial of x over K has coefficients in A.

Proposition 1.5.3: We have $(A \setminus \{0\})^{-1}B = L$.

1.6 Norm and Trace

Assume that L/K is finite separable. Let $\bar{K}$ be an algebraic closure of K.

Definition 1.6.1: For any $x \in L$ we consider the K-linear map $T_x \colon L \to L$, $u \mapsto ux$.

(a) The *norm of x for L/K* is the element $\mathrm{Nm}_{L/K}(x) := \det(T_x) \in K$.

(b) The *trace of x for L/K* is the element $\mathrm{Tr}_{L/K}(x) := \mathrm{tr}(T_x) \in K$.

Proposition 1.6.2: (a) For any $x, y \in L$ we have $\mathrm{Nm}_{L/K}(xy) = \mathrm{Nm}_{L/K}(x) \cdot \mathrm{Nm}_{L/K}(y)$.

(b) The map $\mathrm{Nm}_{L/K}$ induces a homomorphism $L^\times \to K^\times$.

(c) The map $\mathrm{Tr}_{L/K} \colon L \to K$ is K-linear.

Proposition 1.6.3: For any $x \in L$ we have

$$\mathrm{Nm}_{L/K}(x) \;=\; \prod_{\sigma \in \mathrm{Hom}_K(L,\bar{K})} \sigma(x) \qquad \text{and} \qquad \mathrm{Tr}_{L/K}(x) \;=\; \sum_{\sigma \in \mathrm{Hom}_K(L,\bar{K})} \sigma(x).$$

Proposition 1.6.4: The map $\mathrm{Tr}_{L/K} \colon L \to K$ is non-zero.

Proposition 1.6.5: For any two finite separable field extensions $M/L/K$ we have:

(a) $\mathrm{Nm}_{L/K} \circ \mathrm{Nm}_{M/L} = \mathrm{Nm}_{M/K}$.

(b) $\mathrm{Tr}_{L/K} \circ \mathrm{Tr}_{M/L} = \mathrm{Tr}_{M/K}$.

Proposition 1.6.6: For any $x \in B$ we have:

(a) $\mathrm{Nm}_{L/K}(x) \in A$.

(b) $\mathrm{Nm}_{L/K}(x) \in A^\times$ if and only if $x \in B^\times$.

(c) $\mathrm{Tr}_{L/K}(x) \in A$.

1.7 Discriminant

Proposition 1.7.1: The map

$$L \times L \longrightarrow K, \quad (x,y) \mapsto \mathrm{Tr}_{L/K}(xy)$$

is a non-degenerate symmetric K-bilinear form.

Lemma 1.7.2: Write $\mathrm{Hom}_K(L,\bar{K}) = \{\sigma_1, \ldots, \sigma_n\}$ with $[L/K] = n$ and consider the matrix $T := (\sigma_i(b_j))_{i,j=1,\ldots,n}$. Then

$$T^T \cdot T \;=\; \big(\mathrm{Tr}_{L/K}(b_i b_j)\big)_{i,j=1,\ldots,n}.$$

Definition 1.7.3: The *discriminant* of any ordered basis $(b_1, \ldots, b_n)$ of L over K is the determinant of the associated *Gram matrix*

$$\mathrm{disc}(b_1, \ldots, b_n) \;:=\; \det\big(\mathrm{Tr}_{L/K}(b_i b_j)\big)_{i,j=1,\ldots,n} \;=\; \det(T)^2 \;\in\; K.$$

Proposition 1.7.4: If $L = K(b)$ and $n = [L/K]$, then $\mathrm{disc}(1, b, \ldots, b^{n-1})$ is the discriminant of the minimal polynomial of b over K.

Proposition 1.7.5: (a) We have $\mathrm{disc}(b_1, \ldots, b_n) \in K^\times$.

(b) If $b_1, \ldots, b_n \in B$, then $\mathrm{disc}(b_1, \ldots, b_n) \in A \smallsetminus \{0\}$ and

$$B \subset \frac{1}{\mathrm{disc}(b_1, \ldots, b_n)} \cdot \big(Ab_1 + \ldots + Ab_n\big).$$

Proposition 1.7.6: If A is a principal ideal domain, then:

(a) B is a free A-module of rank $[L/K]$.

(b) For any basis $(b_1, \ldots, b_n)$ of B over A, the number $\mathrm{disc}(b_1, \ldots, b_n)$ is independent of the basis up to the square of an element of $A^\times$.

Definition 1.7.7: This number is called the *discriminant of B over A* or *of L over K* and is denoted $\mathrm{disc}_{B/A}$ or $\mathrm{disc}_{L/K}$.

1.8 Linearly disjoint extensions

Definition 1.8.1: Two finite separable field extensions $L, L'/K$ are called *linearly disjoint* if $L \otimes_K L'$ is a field.

Proposition 1.8.2: For any two finite separable field extensions $L, L'/K$ within a common overfield M the following statements are equivalent:

(a) L and L' are linearly disjoint over K.

(b) $[LL'/K] = [L/K] \cdot [L'/K]$

(c) $[LL'/L] = [L'/K]$

(d) $[LL'/L'] = [L/K]$

If at least one of L/K and L'/K is galois, they are also equivalent to

(e) $L \cap L' = K$.

Theorem 1.8.3: Consider linearly disjoint finite separable field extensions $L, L'/K$. Assume that A is a principal ideal domain and that $d := \mathrm{disc}_{L/K}$ and $d' := \mathrm{disc}_{L'/K}$ are relatively prime in A. Let $B, B', \tilde{B}$ be the integral closures of A in L, L', LL'. Then:

(a) $B \otimes_A B' \xrightarrow{\sim} \tilde{B}$.

(b) $\mathrm{disc}_{LL'/K} = d^{[L'/K]} \cdot d'^{[L/K]}$ up to the square of a unit in A.

1.9 Dedekind Rings

Definition 1.9.1: (a) A ring A is *noetherian* if every ideal is finitely generated.

(b) An integral domain A has *Krull dimension* 1 if it is not a field and every non-zero prime ideal is a maximal ideal.

(c) A noetherien normal integral domain of Krull dimension 1 is called a *Dedekind ring.*

Proposition 1.9.2: Any principal ideal domain that is not a field is a Dedekind ring.

Examples 1.9.3: Take $A = \mathbb{Z}$ or $A = \mathbb{Z}[i]$ or $A = k[t]$ or $A = k[[t]]$ for a field k.

In the following we assume that $A \subset K$ is Dedekind and that $B \subset L$ is as above.

Proposition 1.9.4: (a) For every multiplicative subset $S \subset A$ the ring $S^{-1}A$ is Dedekind or a field.

(b) For every prime ideal $0 \neq \mathfrak{p} \subset A$ the localization $A_\mathfrak{p}$ is a discrete valuation ring.

Theorem 1.9.5: The ring B is Dedekind and finitely generated as an A-module.

1.10 Fractional Ideals

Let A be a Dedekind ring with quotient field K.

Definition 1.10.1:

(a) A non-zero finitely generated A-submodule of K is called a *fractional ideal of A.*

(b) A fractional ideal of the form $(x) := Ax$ for some $x \in K^\times$ is called *principal.*

(c) The *product* of two fractional ideals $\mathfrak{a}, \mathfrak{b}$ is defined as

$$\mathfrak{a}\mathfrak{b} \;:=\; \big\{ \textstyle\sum_{i=1}^r a_i b_i \;\big|\; r \geqslant 0,\; a_i \in \mathfrak{a},\; b_i \in \mathfrak{b} \big\}.$$

(d) The *inverse* of a fractional ideal $\mathfrak{a}$ is defined as

$$\mathfrak{a}^{-1} \;=\; \big\{ x \in K \;\big|\; x \cdot \mathfrak{a} \subset A \big\}.$$

Proposition 1.10.2: For any fractional ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we have:

(a) There exist $a, b \in A \smallsetminus \{0\}$ with $(a) \subset \mathfrak{a} \subset (\frac{1}{b})$.

(b) $\mathfrak{a}\mathfrak{b}$ and $\mathfrak{a}^{-1}$ are fractional ideals.

(c) $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ and $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$ and $(1)\mathfrak{a} = \mathfrak{a}$.

(d) $\mathfrak{a} \subset A$ if and only if $A \subset \mathfrak{a}^{-1}$.

Lemma 1.10.3: For every non-zero ideal $\mathfrak{a} \subset A$ there exist an integer $r \geqslant 0$ and maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset \mathfrak{a}$.

Lemma 1.10.4: For every maximal ideal $\mathfrak{p} \subset A$ and every fractional ideal $\mathfrak{a}$ we have

(a) $A \subsetneqq \mathfrak{p}^{-1}$.

(b) $\mathfrak{a} \subsetneqq \mathfrak{p}^{-1}\mathfrak{a}$.

(c) $\mathfrak{p}^{-1}\mathfrak{p} = (1)$.

Theorem 1.10.5: Any non-zero ideal of A is a product of maximal ideals and the factors are unique up to permutation. *(Unique factorization of ideals)*

Theorem 1.10.6: (a) The set J_A of fractional ideals is an abelian group with the above product and inverse and the unit element $(1) = A$.

(b) The group J_A is the free abelian group with basis the maximal ideals of A.

1.11 Ideals

Consider any non-zero ideals $\mathfrak{a}, \mathfrak{b} \subset A$.

Definition 1.11.1: We write $\mathfrak{b}|\mathfrak{a}$ and say that $\mathfrak{b}$ *divides* $\mathfrak{a}$ if and only if $\mathfrak{a} \subset \mathfrak{b}$.

Proposition 1.11.2: For any $a, b \in A \smallsetminus \{0\}$ we have $b|a$ if and only if $(b)|(a)$.

Proposition 1.11.3: We have $\mathfrak{b}|\mathfrak{a}$ if and only if there is a non-zero ideal $\mathfrak{c} \subset A$ with $\mathfrak{b}\mathfrak{c} = \mathfrak{a}$.

Definition 1.11.4: Ideals $\mathfrak{a}, \mathfrak{b} \subset A$ with $\mathfrak{a} + \mathfrak{b} = A$ are called *coprime.*

Proposition 1.11.5: For any non-zero ideals $\mathfrak{a}, \mathfrak{b} \subset A$ the following are equivalent:

(a) $\mathfrak{a}$ and $\mathfrak{b}$ are coprime.

(b) Their factorizations in maximal ideals do not have a common factor.

(c) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

Chinese Remainder Theorem 1.11.6: For any pairwise coprime ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \subset A$ we have a ring isomorphism

$$\begin{aligned} A/\mathfrak{a}_1 \cdots \mathfrak{a}_r &\xrightarrow{\sim} A/\mathfrak{a}_1 \times \ldots \times A/\mathfrak{a}_r, \\ a + \mathfrak{a}_1 \cdots \mathfrak{a}_r &\longmapsto (a + \mathfrak{a}_1, \ldots, a + \mathfrak{a}_r). \end{aligned}$$

Proposition 1.11.7: For any fractional ideals $\mathfrak{a} \subset \mathfrak{b}$ there exists $b \in \mathfrak{b}$ with $\mathfrak{b} = \mathfrak{a} + (b)$.

Proposition 1.11.8: Every fractional ideal of A is generated by 2 elements.

Proposition 1.11.9: For any non-zero ideal $\mathfrak{a}$ and any fractional ideal $\mathfrak{b}$ of A there exists an isomorphism of A-modules $A/\mathfrak{a} \cong \mathfrak{b}/\mathfrak{a}\mathfrak{b}$.

1.12 Ideal class group

Definition 1.12.1: The factor group

$$\mathrm{Cl}(A) \ := \ \big\{\text{fractional ideals}\big\} \ / \ \big\{\text{principal ideals}\big\}$$

is called the *ideal class group of A*. Its order $h(A) := |\,\mathrm{Cl}(A)|$ is called the *class number of A*.

Proposition 1.12.2: Any ideal class is represented by a non-zero ideal of A.

Proposition 1.12.3: There is a fundamental exact sequence

$$1 \longrightarrow A^\times \longrightarrow K^\times \longrightarrow J_A \longrightarrow \mathrm{Cl}(A) \longrightarrow 1.$$

2 Minkowski's lattice theory

2.1 Lattices

Fix a finite dimensional $\mathbb{R}$-vector space V.

Proposition 2.1.1: There exists a unique topology on V such that for any basis $v_1, \ldots, v_n$ of V the isomorphism $\mathbb{R}^n \to V$, $(x_i)_i \mapsto \sum_{i=1}^n x_i v_i$ is a homeomorphism.

Definition 2.1.2: A subset $X \subset V$ is called ...

(a) ... *bounded* if and only if the corresponding subset of $\mathbb{R}^n$ is bounded.

(b) ... *discrete* if and only if the corresponding subset of $\mathbb{R}^n$ is discrete, that is, if its intersection with any bounded subset is finite.

Now we are interested in an (additive) subgroup $\Gamma \subset V$.

Definition-Proposition 2.1.3: The following are equivalent:

(a) Γ is discrete.

(b) $\Gamma = \bigoplus_{i=1}^m \mathbb{Z}v_i$ for $\mathbb{R}$-linearly independent elements $v_1, \ldots, v_m$.

Such a subgroup is called a *lattice.*

Definition-Proposition 2.1.4: The following are equivalent:

(a) Γ is discrete and there exists a bounded subset $\Phi \subset V$ such that $\Gamma + \Phi = V$.

(b) Γ is discrete and V/Γ is compact.

(c) $\Gamma = \bigoplus_{i=1}^n \mathbb{Z}v_i$ for an $\mathbb{R}$-basis $v_1, \ldots, v_n$ of V.

Such a subgroup is called a *complete lattice.*

In the following we consider a lattice $\Gamma \subset V$.

Definition 2.1.5: Any measurable subset $\Phi \subset V$ such that $\Phi \to V/\Gamma$ is bijective is called a *fundamental domain for* Γ. (With respect to the measure from §2.2.)

Example 2.1.6: If $\Gamma = \bigoplus_{i=1}^n \mathbb{Z}v_i$ for an $\mathbb{R}$-basis $v_1, \ldots, v_n$ of V, a fundamental domain is:

$$\Phi := \left\{ \textstyle\sum_{i=1}^n x_i v_i \;\middle|\; \forall i\colon\ 0 \leqslant x_i < 1 \right\}.$$

Caution 2.1.7: If $V \neq 0$, there does not exist a compact fundamental domain, because there is a problem with the boundary.

2.2 Volume

Now we fix a scalar product $\langle\ ,\ \rangle$ on V.

Proposition 2.2.1: (a) There exists a unique Lebesgue measure $d\mathrm{vol}$ on V such that for any measurable function f on V and any orthonormal basis $(e_1, \ldots, e_n)$ of V we have

$$\int_V f(v)\ d\mathrm{vol}(v) \ = \ \int_{\mathbb{R}^n} f\big(\textstyle\sum_{i=1}^n x_i e_i\big)\, dx_1 \ldots dx_n.$$

(b) For any $\mathbb{R}$-basis $(v_1, \ldots, v_n)$ of V we then have

$$\mathrm{vol}\big(\{\textstyle\sum_{i=1}^n x_i v_i \bigm| \forall i\colon\ 0 \leqslant x_i < 1\}\big) \ = \ \sqrt{\det\big(\langle v_i, v_j\rangle\big)_{i,j=1}^n}$$

and

$$\int_V f(v)\ d\mathrm{vol}(v) \ = \ \int_{\mathbb{R}^n} f\big(\textstyle\sum_{i=1}^n y_i v_i\big)\, dy_1 \ldots dy_n \cdot \sqrt{\det\big(\langle v_i, v_j\rangle\big)_{i,j=1}^n}.$$

Definition-Proposition 2.2.2: Consider any fundamental domain $\Phi \subset V$.

(a) For any measurable function f on V/Γ this integral is independent of Φ:

$$\int_{V/\Gamma} f(\bar v)\ d\mathrm{vol}(\bar v) \ := \ \int_\Phi f(v+\Gamma)\ d\mathrm{vol}(v).$$

(b) In particular we obtain

$$\mathrm{vol}(V/\Gamma) \ := \ \int_{V/\Gamma} 1\ d\mathrm{vol}(\bar v) \ = \ \mathrm{vol}(\Phi).$$

Fact 2.2.3: We have $\mathrm{vol}(V/\Gamma) < \infty$ if and only if Γ is a complete lattice.

2.3 Lattice Point Theorem

Let Γ be a complete lattice in a finite dimensional euclidean vector space V.

Definition 2.3.1: A subset $X \subset V$ is *centrally symmetric* if and only if

$$X = -X := \{-x \mid x \in X\}.$$

Theorem 2.3.2: Let $X \subset V$ be a centrally symmetric convex subset which satisfies

$$\mathrm{vol}(X) \ > \ 2^{\dim(V)} \cdot \mathrm{vol}(V/\Gamma).$$

Then $X \cap \Gamma$ contains a non-zero element.

Remark 2.3.3: The theorem is sharp. For example if $V = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$ and $X =]-1,1[^n$, then we have $\mathrm{vol}(X) = 2^{\dim(V)} \cdot \mathrm{vol}(V/\Gamma)$ and $X \cap \Gamma = \{0\}$.

Application 2.3.4: An n-dimensional ball B_r of radius r has volume

$$\mathrm{vol}(B_r) \ = \ \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot r^n.$$

Therefore the smallest non-zero vector in Γ has length

$$\leqslant \ \frac{2}{\sqrt{\pi}} \cdot \sqrt[n]{\mathrm{vol}(V/\Gamma) \cdot \Gamma(\tfrac{n}{2}+1)}.$$

More generally, for every k one can bound the combined lengths of k linearly independent vectors in Γ using *successive minima*.

3 Algebraic integers

3.1 Number fields

Definition 3.1.1: (a) A finite field extension $K/\mathbb{Q}$ is called an *(algebraic) number field.*

(b) A number field of degree $2, 3, 4, 5$,... is called *quadratic, cubic, quartic, quintic,...*

(c) The integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in K is called the ring of *algebraic integers in K.*

In the rest of this chapter we fix such K and $\mathcal{O}_K$ and abbreviate $n := [K/\mathbb{Q}]$.

Proposition 3.1.2: (a) The ring $\mathcal{O}_K$ is Dedekind.

(c) $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n.

(b) Any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n.

3.2 Absolute discriminant

Proposition 3.2.1: (a) For any $\mathbb{Z}$-submodule $\Gamma \subset K$ of rank n with an ordered $\mathbb{Z}$-basis $(x_1, \ldots, x_n)$ the following value depends only on Γ:

$$\operatorname{disc}(\Gamma) := \operatorname{disc}(x_1, \ldots, x_n) \in \mathbb{Q}^\times.$$

(b) For any two $\mathbb{Z}$-submodules $\Gamma \subset \Gamma' \subset K$ of rank n the index $[\Gamma' : \Gamma]$ is finite and we have

$$\operatorname{disc}(\Gamma) = [\Gamma' : \Gamma]^2 \cdot \operatorname{disc}(\Gamma').$$

(c) For any $\mathbb{Z}$-submodule $\Gamma \subset \mathcal{O}_K$ of rank n we have $\operatorname{disc}(\Gamma) \in \mathbb{Z} \smallsetminus \{0\}$.

Definition 3.2.2: The number

$$d_K \ := \ \operatorname{disc}(\mathcal{O}_K) \ \in \ \mathbb{Z} \smallsetminus \{0\}$$

is called the *discriminant of $\mathcal{O}_K$ or of K.*

Corollary 3.2.3: If there exist $a_1, \ldots, a_n \in \mathcal{O}_K$ such that $\operatorname{disc}(a_1, \ldots, a_n)$ is squarefree, then

$$\mathcal{O}_K = \mathbb{Z}a_1 \oplus \ldots \oplus \mathbb{Z}a_n.$$

3.3 Absolute norm

Definition 3.3.1: The *absolute norm* of a non-zero ideal $\mathfrak{a} \subset \mathcal{O}_K$ is the index

$$\operatorname{Nm}(\mathfrak{a}) := [\mathcal{O}_K \colon \mathfrak{a}] \in \mathbb{Z}^{\geqslant 1}.$$

Proposition 3.3.2: For any $a \in \mathcal{O}_K \smallsetminus \{0\}$ we have $\mathrm{Nm}((a)) = |\,\mathrm{Nm}_{K/\mathbb{Q}}(a)|$.

Proposition 3.3.3: For any integer $N \geqslant 1$ there exist only finitely many non-zero ideals $\mathfrak{a} \subset \mathcal{O}_K$ with $\mathrm{Nm}(\mathfrak{a}) \leqslant N$.

Proposition 3.3.4: For any two non-zero ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_K$ we have

$$\mathrm{Nm}(\mathfrak{a}\mathfrak{b}) = \mathrm{Nm}(\mathfrak{a}) \cdot \mathrm{Nm}(\mathfrak{b}).$$

Let J_K denote the group of fractional ideals of $\mathcal{O}_K$.

Corollary 3.3.5: The absolute norm extends to a unique homomorphism

$$\mathrm{Nm}\colon J_K \longrightarrow (\mathbb{Q}^{>0}, \cdot).$$

3.4 Real and complex embeddings

Throughout the following we abbreviate $\Sigma := \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ and set

$$\begin{aligned} r &:= \text{ the number of } \sigma \in \Sigma \text{ with } \sigma(K) \subset \mathbb{R}, \\ s &:= \text{ the number of } \sigma \in \Sigma \text{ with } \sigma(K) \not\subset \mathbb{R}, \text{ up to complex conjugation.} \end{aligned}$$

Proposition 3.4.1: We have $r + 2s = n$.

Proposition 3.4.2: We have ring isomorphisms

$$\begin{array}{ccl} K \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\quad\sim\quad} & K_{\mathbb{C}} := \prod_{\sigma\in\Sigma} \mathbb{C}, \\ \cup & & \cup \\ K \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\quad\sim\quad} & K_{\mathbb{R}} := \big\{ (z_\sigma)_\sigma \in K_{\mathbb{C}} \,\big|\, \forall \sigma \in \Sigma\colon z_{\bar\sigma} = \bar z_\sigma \big\}. \\ x \otimes z & \longmapsto & (\sigma(x)z)_\sigma. \end{array}$$

The map $x \mapsto x \otimes 1$ induces an embdding $j\colon K \hookrightarrow K_{\mathbb{R}}$.

Proposition 3.4.3: For every fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ the image $j(\mathfrak{a})$ is a complete lattice in $K_{\mathbb{R}}$.

To describe this with more explicit coordinates we let $\sigma_1, \ldots, \sigma_r$ be the real embeddings and $\sigma_{r+1}, \ldots, \sigma_n$ the non-real embeddings such that $\bar\sigma_{r+j} = \sigma_{r+j+s}$ for all $1 \leqslant j \leqslant s$.

Proposition 3.4.4: We have an isomorphism of $\mathbb{R}$-vector spaces

$$K_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n,\ (z_\sigma)_\sigma \longmapsto \big(z_{\sigma_1}, \ldots, z_{\sigma_r}, \mathrm{Re}\, z_{\sigma_{r+1}}, \ldots, \mathrm{Re}\, z_{\sigma_{r+s}}, \mathrm{Im}\, z_{\sigma_{r+1}}, \ldots, \mathrm{Im}\, z_{\sigma_{r+s}}\big).$$

3.5 Quadratic number fields

Proposition 3.5.1: The quadratic number fields are precisely the splitting fields of the poiynomials $X^2 - d$ for all squarefree integers $d \in \mathbb{Z} \smallsetminus \{0,1\}$.

Convention 3.5.2: For any positive integer d we let $\sqrt{d}$ be the positive real square root of d. For any negative integer d we uncanonically *choose* a square root $\sqrt{d}$ in $i\mathbb{R}$.

Proposition 3.5.2: For d as above and $K = \mathbb{Q}(\sqrt{d})$ we have

$$\mathcal{O}_K \;=\; \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2,3 \bmod (4), \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \bmod (4) \end{cases}$$

and

$$d_K \;=\; \begin{cases} 4d & \text{if } d \equiv 2,3 \bmod (4), \\ d & \text{if } d \equiv 1 \bmod (4) \end{cases}$$

Corollary 3.5.4: The integer d is uniquely determined by K, namely as the squarefree part of d_K.

Remark 3.5.5: The possible discriminants of quadratic number fields are sometimes called *fundamental discriminants.* As the discriminant is somewhat more canonically associated to K than the number d, some authors prefer to write $K = \mathbb{Q}(\sqrt{d_K})$.

Definition 3.5.6: We have the following cases:

(a) If $d > 0$, there exist precisely two distinct embeddings $\sigma_1, \sigma_2 \colon K \hookrightarrow \mathbb{R}$ and we call K *real quadratic.* In this case we obtain a natural embedding

$$(\sigma_1, \sigma_2) \colon K \lhook\joinrel\longrightarrow \mathbb{R}^2.$$

(b) If $d < 0$, there exist precisely two distinct embeddings $\sigma, \bar{\sigma} \colon K \hookrightarrow \mathbb{C}$ that are conjugate under complex conjugation, and we call K *imaginary quadratic.* In this case we obtain a natural embedding

$$\sigma \colon K \lhook\joinrel\longrightarrow \mathbb{C}.$$

3.6 Cyclotomic fields

Fix an integer $n \geqslant 1$.

Definition 3.6.1: (a) An element $\zeta \in \mathbb{C}$ with $\zeta^n = 1$ is called an *n-th root of unity.*

(b) An element $\zeta \in \mathbb{C}^\times$ of precise order n is called a *primitive n-th root of unity.*

Proposition 3.6.2: The n-th roots of unity form a cyclic subgroup $\mu_n \subset \mathbb{C}^\times$, which is generated by any primitive n-th root of unity, for instance by $e^{\frac{2\pi i}{n}}$.

For the following we fix a primitive n-th root of unity ζ and set $K := \mathbb{Q}(\mu_n) = \mathbb{Q}(\zeta)$.

Proposition 3.6.3: (a) An integral power ζ^a has order n if and only if $\gcd(a,n)=1$.

(b) If $n \geqslant 2$, then for any such a we have $\frac{1-\zeta^a}{1-\zeta} \in \mathcal{O}_K^\times$. *(Cyclotomic units)*

Definition 3.6.4: The *n-th cyclotomic polynomial* Φ_n is the monic polynomial of degree $\varphi(n) := |(\mathbb{Z}/n\mathbb{Z})^\times|$ with the primitive n-th roots of unity as simple roots.

Theorem 3.6.5: The polynomial Φ_n is an irreducible element of $\mathbb{Z}[X]$.

Theorem 3.6.6: The extension $K/\mathbb{Q}$ is finite galois of degree $\varphi(n)$ and there is a natural isomorphism $e\colon \operatorname{Gal}(K/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times$ with the property

$$\forall \gamma \in \operatorname{Gal}(K/\mathbb{Q})\colon\ \gamma(\zeta) = \zeta^{e(\gamma)}.$$

Theorem 3.6.7: If $n = \ell^\nu$ for a prime ℓ and an integer $\nu \geqslant 1$, then:

(a) We have $\Phi_{\ell^\nu}(X) = \sum_{i=0}^{\ell-1} X^{i\ell^{\nu-1}}$.

(b) The ideal $(1-\zeta)$ of $\mathcal{O}_K$ satisfies $(1-\zeta)^{\ell^{\nu-1}(\ell-1)} = (\ell)$.

(c) The ideal $(1-\zeta)$ is the unique prime ideal of $\mathcal{O}_K$ above $(\ell) \subset \mathbb{Z}$ and has the residue field $\mathcal{O}_K/(1-\zeta) \cong \mathbb{F}_\ell$.

(d) $\mathcal{O}_K = \mathbb{Z}[\zeta] \cong \mathbb{Z}[X]/(\Phi_{\ell^\nu})$.

(e) $\operatorname{disc}(\mathcal{O}_K) = \pm \ell^{\ell^{\nu-1}(\nu\ell-\nu-1)}$.

Theorem 3.6.8: For arbitrary n we have:

(a) $\mathcal{O}_K = \mathbb{Z}[\zeta]$.

(b) The discriminant $\operatorname{disc}(\mathcal{O}_K) \in \mathbb{Z}$ is divisible precisely by the primes dividing n.

3.7 Quadratic Reciprocity

Fix an odd prime ℓ and set $K := \mathbb{Q}(\mu_\ell)$ and $\zeta := e^{\frac{2\pi i}{\ell}}$.

Definition 3.7.1: The *Legendre symbol* of an integer a with respect to ℓ is

$$\Big(\frac{a}{\ell}\Big) := \begin{cases} \ \ 0 & \text{if } a \equiv 0 \bmod (\ell), \\ +1 & \text{if } a \equiv b^2 \bmod (\ell) \text{ for some } b \in \mathbb{Z} \smallsetminus \ell\mathbb{Z}, \\ -1 & \text{otherwise.} \end{cases}$$

In the first two cases a is called a *quadratic residue*, otherwise a *quadratic non-residue modulo* (ℓ).

Proposition 3.7.2: For any integers a, b we have:

(a) $(\frac{a}{\ell}) = (\frac{b}{\ell})$ whenever $a \equiv b \bmod (\ell)$.

(b) $(\frac{a}{\ell}) \equiv a^{\frac{\ell-1}{2}} \bmod (\ell)$.

(c) $(\frac{ab}{\ell}) = (\frac{a}{\ell})(\frac{b}{\ell})$.

(d) $(\frac{-1}{\ell}) = (-1)^{\frac{\ell-1}{2}}$.

Definition 3.7.3: The *Gauss sum* associated to the prime ℓ is $g_\ell := \sum_{a=1}^{\ell-1} \big(\frac{a}{\ell}\big) \cdot \zeta^a$.

Proposition 3.7.4: The Gauss sum satisfies $g_\ell^2 = \ell^* := (-1)^{\frac{\ell-1}{2}}\ell$.

Proposition 3.7.5: The unique subfield of K of degree 2 over $\mathbb{Q}$ is $K' := \mathbb{Q}(\sqrt{\ell^*}\,)$.

Proposition 3.7.6: For any distinct odd primes ℓ, p we have $\big(\frac{\ell^*}{p}\big) = \big(\frac{p}{\ell}\big)$.

Theorem 3.7.7: *(Gauss Quadratic Reciprocity Law)*

(a) For any distinct odd primes ℓ, p we have $\big(\frac{\ell}{p}\big)\big(\frac{p}{\ell}\big) = (-1)^{\frac{(p-1)(\ell-1)}{4}}$.

(b) For any odd prime ℓ we have $\big(\frac{-1}{\ell}\big) = (-1)^{\frac{\ell-1}{2}}$. *(First supplement)*

(c) For any odd prime ℓ we have $\big(\frac{2}{\ell}\big) = (-1)^{\frac{\ell^2-1}{8}}$. *(Second supplement)*

4 Additive Minkowski theory

4.1 Euclidean embedding

We endow $K_{\mathbb{C}} := \mathbb{C}^{\Sigma}$ with the standard hermitian scalar product

$$\big\langle (z_\sigma)_\sigma, (w_\sigma)_\sigma \big\rangle \;:=\; \sum_{\sigma\in\Sigma} \bar{z}_\sigma w_\sigma.$$

Proposition 4.1.1: Its restriction to $K_{\mathbb{R}} \times K_{\mathbb{R}}$ has values in $\mathbb{R}$ and turns $K_{\mathbb{R}}$ into a euclidean vector space.

Proposition 4.1.2: Under the isomorphism of Proposition 3.4.4 this scalar product on $K_{\mathbb{R}}$ corresponds to the following scalar product on $\mathbb{R}^n$:

$$\big\langle (x_j)_j, (y_j)_j \big\rangle \;:=\; \sum_{j=1}^{r} x_j y_j + \sum_{j=r+1}^{n} 2x_j y_j.$$

4.2 Lattice bounds

Proposition 4.2.1: For any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ we have

$$\operatorname{vol}(K_{\mathbb{R}}/j(\mathfrak{a})) \;=\; \sqrt{|\operatorname{disc}(\mathfrak{a})|} \;=\; \operatorname{Nm}(\mathfrak{a}) \cdot \sqrt{|d_K|}.$$

Theorem 4.2.2: Consider a fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ and positive real numbers c_σ for all $\sigma \in \Sigma$ such that $c_{\bar\sigma} = c_\sigma$ and

$$\prod_{\sigma\in\Sigma} c_\sigma \;>\; (\tfrac{2}{\pi})^s \cdot \sqrt{|d_K|} \cdot \operatorname{Nm}(\mathfrak{a}).$$

Then there exists an element $a \in \mathfrak{a} \smallsetminus \{0\}$ with the property

$$\forall \sigma \in \Sigma\colon\;\; |\sigma(a)| < c_\sigma.$$

4.3 Finiteness of the class group

Theorem 4.3.1: For any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ there exists an element $a \in \mathfrak{a} \smallsetminus \{0\}$ with

$$|\operatorname{Nm}_{K/\mathbb{Q}}(a)| \;\leqslant\; (\tfrac{2}{\pi})^s \cdot \sqrt{|d_K|} \cdot \operatorname{Nm}(\mathfrak{a}).$$

Proposition 4.3.2: Every ideal class in $\operatorname{Cl}(\mathcal{O}_K)$ contains an ideal $\mathfrak{a} \subset \mathcal{O}_K$ with

$$\operatorname{Nm}(\mathfrak{a}) \;\leqslant\; (\tfrac{2}{\pi})^s \cdot \sqrt{|d_K|}.$$

Theorem 4.3.3: The class group $\operatorname{Cl}(\mathcal{O}_K)$ is finite.

4.4 Discriminant bounds

Theorem 4.4.1: For any n and c there exist at most finitely many number fields $K/\mathbb{Q}$ of degree n and with $|d_K| \leqslant c$.

Theorem 4.4.2: For any number field K of degree n over $\mathbb{Q}$ we have

$$\sqrt{|d_K|} \;\geqslant\; \frac{n^n}{n!} \cdot \Big(\frac{\pi}{4}\Big)^{n/2}.$$

Theorem 4.4.3: *(Hermite)* For any c there exist at most finitely many number fields $K/\mathbb{Q}$ with $|d_K| \leqslant c$.

Theorem 4.4.4: *(Minkowski)* For any number field $K \neq \mathbb{Q}$ we have $|d_K| > 1$.

5 Multiplicative Minkowski theory

5.1 Roots of unity

Lemma 5.1.1: We have a short exact sequence

$$\begin{array}{ccccccccc} 1 \longrightarrow (S^1)^\Sigma \longrightarrow K_{\mathbb{C}}^\times & = & (\mathbb{C}^\times)^\Sigma & \xrightarrow{\ \ell\ } & \mathbb{R}^\Sigma \longrightarrow 0, \\ & & (z_\sigma)_\sigma & \longmapsto & (\log|z_\sigma|)_\sigma. \end{array}$$

Set $\Gamma := \ell(\mathcal{O}_K^\times)$ and let $\mu(K)$ denote the group of elements of finite order in $K^\times$.

Proposition 5.1.2: The group $\mu(K)$ is a finite subgroup of $\mathcal{O}_K^\times$ and we have a short exact sequence

$$1 \longrightarrow \mu(K) \longrightarrow \mathcal{O}_K^\times \longrightarrow \Gamma \longrightarrow 0.$$

Proposition 5.1.3: The group $\mu(K)$ is cyclic of even order.

Example 5.1.4: For any squarefree $d \in \mathbb{Z} \smallsetminus \{1\}$ we have

$$\mu(\mathbb{Q}(\sqrt{d}\,)) \;=\; \begin{cases} \text{cyclic of order 6 if } d = -3, \\ \text{cyclic of order 4 if } d = -1, \\ \text{cyclic of order 2 otherwise.} \end{cases}$$

5.2 Units

Lemma 5.2.1: The group Γ is a lattice in $\mathbb{R}^\Sigma$.

Consider the homomorphisms

$$\begin{array}{lrl} \mathrm{Nm}\colon & K_{\mathbb{C}}^\times = (\mathbb{C}^\times)^\Sigma \longrightarrow \mathbb{C}^\times, & (z_\sigma)_\sigma \longmapsto \prod_{\sigma\in\Sigma} z_\sigma \\ \mathrm{Tr}\colon & (\mathbb{R}^\times)^\Sigma \longrightarrow \mathbb{R}, & (t_\sigma)_\sigma \longmapsto \sum_{\sigma\in\Sigma} t_\sigma \end{array}$$

Lemma 5.2.2: We have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_K^\times \arrow[r, hook] \arrow[d, "\mathrm{Nm}"'] & K^\times \arrow[r, hook, "j"] \arrow[d, "\mathrm{Nm}"'] & (K_{\mathbb{C}})^\times \arrow[r, two heads, "\ell"] \arrow[d, "\mathrm{Nm}"'] & \mathbb{R}^\Sigma \arrow[d, "\mathrm{Tr}"'] \\ \{\pm1\} \arrow[r, hook] & \mathbb{Q}^\times \arrow[r, hook] & \mathbb{C}^\times \arrow[r, two heads, "\log|\ |"] & \mathbb{R} \end{tikzcd}$$

Consider the $\mathbb{R}$-subspaces

$$\begin{array}{lcl} (\mathbb{R}^\Sigma)^+ & := & \big\{(t_\sigma)_\sigma \in \mathbb{R}^\Sigma \;\big|\; \forall\sigma\colon t_{\bar\sigma} = t_\sigma\big\}, \\ H & := & \ker\big(\mathrm{Tr}\colon (\mathbb{R}^\Sigma)^+ \to \mathbb{R}\big). \end{array}$$

Lemma 5.2.3: We have $\Gamma \subset H$ and $\dim_{\mathbb{R}}(H) = r+s-1$.

5.3 Dirichlet's unit theorem

Theorem 5.3.1: The group Γ is a complete lattice in H.

Theorem 5.3.2: The group $\mathcal{O}_K^\times$ is isomorphic to $\mu(K) \times \mathbb{Z}^{r+s-1}$.

Caution 5.3.3: The isomorphism is uncanonical.

Corollary 5.3.4: The group $\mathcal{O}_K^\times$ is finite if and only if K is $\mathbb{Q}$ or imaginary quadratic.

Corollary 5.3.5: The group $\mathcal{O}_K^\times$ has $\mathbb{Z}$-rank 1 if and only if $(r,s) \in \big\{(2,0),(1,1),(0,2)\big\}$. In that case we have

$$\mathcal{O}_K^\times \;=\; \mu(K) \times \varepsilon^{\mathbb{Z}}$$

for some unit ε of infinite order.

Definition 5.3.6: Any choice of such ε is then called a *fundamental unit*.

5.4 The real quadratic case

Suppose that $K = \mathbb{Q}(\sqrt{d})$ for a squarefree $d > 1$ and choose an embedding $K \hookrightarrow \mathbb{R}$.

Fact 5.4.1: There is a unique choice of fundamental unit $\varepsilon > 1$.

Proposition 5.4.2: If $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$, then

(a) $\mathcal{O}_K^\times = \big\{a + b\sqrt{d} \;\big|\; a,b \in \mathbb{Z},\ a^2 - b^2 d = \pm 1\big\}$.

(b) $\mathcal{O}_K^\times \cap \mathbb{R}^{>1} = \big\{a + b\sqrt{d} \;\big|\; a,b \in \mathbb{Z},\ a^2 - b^2 d = \pm 1,\ a,b > 0\big\}$.

(c) The fundamental unit $\varepsilon > 1$ is the element $a + b\sqrt{d} \in \mathcal{O}_K^\times \cap \mathbb{R}^{>1}$ as in (b) with the smallest value for a, or equivalently for b.

Theorem 5.4.3: For any squarefree integer $d > 1$ there are infinitely many solutions $(a,b) \in \mathbb{Z}^2$ of the diophantine equation $a^2 - b^2 d = 1$.

Remark 5.4.4: The equation $a^2 - b^2 d = -1$ may or may not have a solution $(a,b) \in \mathbb{Z}^2$. But if it has a solution, it has infinitely many.

Proposition 5.4.5: The fundamental unit $\varepsilon > 1$ of K with discriminant D satisfies

$$\varepsilon \;\geqslant\; \frac{\sqrt{D} + \sqrt{D-4}}{2} \;>\; 1.$$

Consequently, if some unit of infinite order $u > 1$ is known, we have $u = \varepsilon^k$ for some $1 \leqslant k \leqslant \log(u)/\log((\sqrt{D} + \sqrt{D-4})/2)$ and one can efficiently find ε.

Remark 5.4.6: One can effectively find ε using continued fractions.

6 Extensions of Dedekind rings

6.1 Modules over Dedekind rings

Let A be a Dedekind ring with quotient field K.

Definition 6.1.1: Consider an A-module M.

(a) An element $m \in M$ is called *torsion* if there exists $a \in A \smallsetminus \{0\}$ such that $am = 0$.

(b) The module M is called *torsion* if every element of M is torsion.

(c) The module M is called *torsion-free* if no non-zero element of M is torsion.

Theorem 6.1.2: Any finitely generated A-module is isomorphic to the direct sum of a torsion module and a torsion-free module.

Theorem 6.1.3: Any non-zero finitely generated torsion-free A-module is isomorphic to $\mathfrak{a} \oplus A^{r-1}$ for a non-zero ideal $\mathfrak{a} \subset A$ and an integer $r \geqslant 1$.

Theorem 6.1.4: Any finitely generated torsion A-module is isomorphic to

(a) $\bigoplus_{i=1}^r A/\mathfrak{p}_i^{e_i}$ for $r \geqslant 0$ and maximal ideals $\mathfrak{p}_i \subset A$ and integral exponents $e_i \geqslant 1$.

(b) $\bigoplus_{i=1}^s A/\mathfrak{a}_i$ for $s \geqslant 0$ and non-zero ideals $\mathfrak{a}_s \subset \ldots \subset \mathfrak{a}_1 \subsetneqq A$.

Proposition 6.1.5: Consider a K-vector space V of finite dimension n and a finitely generated A-submodule $M \subset V$ that generates V over K. Then M is isomorphic to a direct sum of n fractional ideals of A.

Proposition 6.1.6: For any fractional ideals $\mathfrak{a}, \mathfrak{b}$ of A there is a natural isomorphism

$$\mathfrak{b}\mathfrak{a}^{-1} \xrightarrow{\sim} \mathrm{Hom}_A(\mathfrak{a}, \mathfrak{b}), \quad c \mapsto (\varphi_c \colon a \mapsto ca).$$

6.2 Decomposition of prime ideals

For the rest of this chapter we take a finite separable field extension L/K of degree n. Then the integral closure B of A in L is a finitely generated A-module that generates L as a K-vector space and is a Dedekind ring. We abbreviate the residue field at any maximal ideal $\mathfrak{p} \subset A$ by $k(\mathfrak{p}) := A/\mathfrak{p}$, and likewise for any maximal ideal of B. Where applicable we let C be the integral closure of B in a finite separable extension M/L.

Consider a maximal ideal $\mathfrak{p} \subset A$. Throughout the following we impose the

Assumption 6.2.1: The residue field $k(\mathfrak{p})$ is perfect.

Note that $\mathfrak{p}B$ is a non-zero ideal of B and therefore has a unique prime factorization

$$\mathfrak{p}B \;=\; \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_r^{e_r}$$

with distinct maximal ideals $\mathfrak{q}_i \subset B$ and integral exponents $e_i \geqslant 1$.

Proposition 6.2.2: (a) The ideals $\mathfrak{q}_i$ are precisely the prime ideals of B above $\mathfrak{p}$.

(b) For each i the residue field $k(\mathfrak{q}_i)$ is a finite extension of the residue field $k(\mathfrak{p})$.

(c) Letting f_i denote the degree of this residue field extension, we have

$$\sum_{i=1}^{r} e_i f_i = n.$$

Definition 6.2.3:

(a) The number $e_{\mathfrak{q}_i|\mathfrak{p}} := e_i$ is called the *ramification degree of* $\mathfrak{q}_i$ *over* $\mathfrak{p}$.

(b) The number $f_{\mathfrak{q}_i|\mathfrak{p}} := f_i$ is called the *inertia degree of* $\mathfrak{q}_i$ *over* $\mathfrak{p}$.

(c) We call $\mathfrak{q}_i$ *unramified over* $\mathfrak{p}$ if $e_i = 1$.

(d) We call $\mathfrak{q}_i$ *ramified over* $\mathfrak{p}$ if $e_i > 1$.

Definition 6.2.4:

(a) We call $\mathfrak{p}$ *unramified in* B if all $e_i = 1$, that is, if $\mathfrak{p}B = \mathfrak{q}_1 \cdots \mathfrak{q}_r$.

(b) We call $\mathfrak{p}$ *ramified in* B if some $e_i > 1$.

(c) We call $\mathfrak{p}$ *totally split in* B if all $e_i = f_i = 1$, that is, if $r = n$ and $\mathfrak{p}B = \mathfrak{q}_1 \cdots \mathfrak{q}_n$.

(d) We call $\mathfrak{p}$ *totally inert in* B if $r = e_1 = 1$, that is, if $\mathfrak{p}B$ is prime.

(e) We call $\mathfrak{p}$ *totally ramified in* B if $r = f_1 = 1$, that is, if $\mathfrak{p}B = \mathfrak{q}^n$ for a prime $\mathfrak{q} \subset B$.

Proposition 6.2.5: Suppose that $B = A[\beta]$ and let $f \in A[X]$ be the minimal polynomial of β above K. Set $\bar{f} := f \bmod \mathfrak{p}$ and write $\bar{f} = \prod_{i=1}^{r} \bar{f}_i^{e_i}$ with inequivalent irreducible factors $\bar{f}_i \in k(\mathfrak{p})[X]$ and integral exponents $e_i \geqslant 1$. Choose $f_i \in A[X]$ with $\bar{f}_i = f_i \bmod \mathfrak{p}$. Then $\mathfrak{p}B = \prod_{i=1}^{r} \mathfrak{q}_i^{e_i}$ with distinct prime ideals $\mathfrak{q}_i := \mathfrak{p}B + f_i(\beta)B$.

Example 6.2.6: Take $L = \mathbb{Q}(\sqrt{d}\,)$ with $d \in \mathbb{Z} \smallsetminus \{1\}$ squarefree. Then an odd prime p of $\mathbb{Z}$ with

$$\left(\frac{d}{p}\right) = \begin{cases} 0 & \text{is (totally) ramified in } \mathcal{O}_L, \\ 1 & \text{is (totally) decomposed in } \mathcal{O}_L, \\ -1 & \text{is (totally) inert in } \mathcal{O}_L. \end{cases}$$

Proposition 6.2.7: For any a prime $\mathfrak{r} \subset C$ above $\mathfrak{q} \subset B$ above $\mathfrak{p} \subset A$ we have

$$e_{\mathfrak{r}|\mathfrak{p}} = e_{\mathfrak{r}|\mathfrak{q}} \cdot e_{\mathfrak{q}|\mathfrak{p}} \qquad \text{and} \qquad f_{\mathfrak{r}|\mathfrak{p}} = f_{\mathfrak{r}|\mathfrak{q}} \cdot f_{\mathfrak{q}|\mathfrak{p}}.$$

6.3 Decomposition group

From now until §6.5 we assume in addition that L/K is galois with Galois group Γ.

Lemma 6.3.1: For any prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and any ideal $\mathfrak{a}$ of a ring we have

$$\mathfrak{a} \subset \bigcup_{i=1}^{n} \mathfrak{p}_i \quad \Longleftrightarrow \quad \exists\, i\colon \mathfrak{a} \subset \mathfrak{p}_i.$$

Theorem 6.3.2: (a) The group Γ acts on B and on the set of prime ideals of B.

(b) The group Γ acts transitively on the set of prime ideals $\mathfrak{q} \subset B$ above $\mathfrak{p}$.

Definition 6.3.3: The stabilizer of $\mathfrak{q}$ is called the *decomposition group of* $\mathfrak{q}$:

$$\Gamma_{\mathfrak{q}} \;:=\; \big\{\gamma \in \Gamma \;\big|\; \forall x \in \mathfrak{q}\colon {}^{\gamma}x \in \mathfrak{q}\big\}.$$

Proposition 6.3.4:

(a) The numbers $e := e_{\mathfrak{q}|\mathfrak{p}}$ and $f := f_{\mathfrak{q}|\mathfrak{p}}$ depend only on $\mathfrak{p}$.

(b) We have $\mathfrak{p}B = \prod_{[\gamma]\in\Gamma/\Gamma_{\mathfrak{q}}} {}^{\gamma}\mathfrak{q}^{e}$.

(c) We have $n = r \cdot e \cdot f$.

(d) For any $\gamma \in \Gamma$ we have $\Gamma_{{}^{\gamma}\mathfrak{q}} = {}^{\gamma}\Gamma_{\mathfrak{q}}$.

Proposition 6.3.5:

(a) We have $\Gamma_{\mathfrak{q}} = 1$ if and only if $\mathfrak{p}$ is totally split in B.

(b) We have $\Gamma_{\mathfrak{q}} = \Gamma$ if and only if there is a unique prime $\mathfrak{q} \subset B$ above $\mathfrak{p}$.

Proposition 6.3.6: Set $L' := L^{\Gamma_{\mathfrak{q}}}$ and $B' := B \cap L'$ and $\mathfrak{q}' := \mathfrak{q} \cap B'$.

(a) Then $\mathfrak{q}$ is the unique prime of B above $\mathfrak{q}'$ and $\mathfrak{q}'B = \mathfrak{q}^{e}$.

(b) We have $e_{\mathfrak{q}|\mathfrak{q}'} = e$ and $f_{\mathfrak{q}|\mathfrak{q}'} = f$ and $e_{\mathfrak{q}'|\mathfrak{p}} = f_{\mathfrak{q}'|\mathfrak{p}} = 1$.

6.4 Inertia group

Next $\Gamma_{\mathfrak{q}}$ acts on the residue field $k(\mathfrak{q}) := B/\mathfrak{q}$ by a natural homomorphism

$$\Gamma_{\mathfrak{q}} \longrightarrow \mathrm{Aut}(k(\mathfrak{q})/k(\mathfrak{p})).$$

Definition 6.4.1: Its kernel is called the *inertia group of* $\mathfrak{q}$:

$$I_{\mathfrak{q}} \;:=\; \big\{\gamma \in \Gamma \;\big|\; \forall x \in A\colon {}^{\gamma}x \equiv x \bmod \mathfrak{q}\big\}.$$

Proposition 6.4.2: The extension $k(\mathfrak{q})/k(\mathfrak{p})$ is galois and the above homomorphism induces an isomorphism $\Gamma_{\mathfrak{q}}/I_{\mathfrak{q}} \cong \mathrm{Aut}(k(\mathfrak{q})/k(\mathfrak{p}))$.

Proposition 6.4.3: Set $L'' := L^{I_{\mathfrak{q}}}$ and $B'' := B \cap L''$ and $\mathfrak{q}'' := \mathfrak{q} \cap B''$.

(a) Then $\mathfrak{q}'B'' = \mathfrak{q}''$ and $\mathfrak{q}''B = \mathfrak{q}^{e}$.

(b) We have $|I_{\mathfrak{q}}| = e$ and $[\Gamma_{\mathfrak{q}}\colon I_{\mathfrak{q}}] = f$ and $[\Gamma : \Gamma_{\mathfrak{q}}] = r$.

(c) We have $e_{\mathfrak{q}|\mathfrak{q}''} = e$ and $f_{\mathfrak{q}|\mathfrak{q}''} = e_{\mathfrak{q}''|\mathfrak{p}'} = 1$ and $f_{\mathfrak{q}''|\mathfrak{p}'} = f$.

6.5 Frobenius

Keeping L/K galois with group Γ, we now assume that $k(\mathfrak{p})$ is finite. Then $k(\mathfrak{q})/k(\mathfrak{p})$ is finite galois, and its Galois group is generated by the Frobenius automorphism $x \mapsto x^{|k(\mathfrak{p})|}$.

Proposition 6.5.1: (a) There exists $\gamma \in \Gamma_\mathfrak{q}$ that acts on $k(\mathfrak{q})$ through $x \mapsto x^{|k(\mathfrak{p})|}$.

(b) The coset $\gamma I_\mathfrak{q}$ is uniquely determined by $\mathfrak{q}$.

Definition 6.5.2: Any such γ is called a *Frobenius substitution at* $\mathfrak{q}$ and denoted by $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}}$.

Proposition 6.5.3: If $\mathfrak{q}$ is unramified over $\mathfrak{p}$, then in addition:

(a) The element $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}}$ is uniquely determined by $\mathfrak{q}$.

(c) The conjugacy class of $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}}$ in Γ is uniquely determined by $\mathfrak{p}$.

(d) If Γ is abelian, then $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}}$ is uniquely determined by $\mathfrak{p}$.

Caution 6.5.4: Do not confuse the Frobenius substitution $\mathrm{Frob}_{\mathfrak{q}|\mathfrak{p}} \in \Gamma_\mathfrak{q}$ with the Frobenius automorphism $x \mapsto x^{|k(\mathfrak{p})|}$ of $k(\mathfrak{q})$.

Example 6.5.5: Consider the cyclotomic field $L := \mathbb{Q}(\mu_n)$ for $n \not\equiv 2 \bmod (4)$.

(a) A rational prime p is ramified in $\mathcal{O}_L$ if and only if $p|n$.

(b) For any $p \nmid n$ the Frobenius substitution at p corresponds to the residue class of p under the isomorphism $\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.

(c) A rational prime p is totally split in $\mathcal{O}_L$ if and only if $p \equiv 1 \bmod (n)$.

(d) If $n = p^\nu$ for a prime p, then p is totally ramified in $\mathcal{O}_L$.

6.6 Relative norm

Now we return to the situation that L/K is finite separable of degree n.

Definition 6.6.1: The *relative norm* of a fractional ideal $\mathfrak{b}$ of B is the A-submodule

$$\mathrm{Nm}_{L/K}(\mathfrak{b}) \;:=\; \big(\{\mathrm{Nm}_{L/K}(y) \mid y \in \mathfrak{b}\}\big) \;\subset\; K.$$

Proposition 6.6.2:

(a) This is a fractional ideal of A.

(b) If $\mathfrak{b} \subset B$ then $\mathrm{Nm}_{L/K}(\mathfrak{b}) \subset \mathfrak{b} \cap A$.

(c) For any $y \in L^\times$ we have $\mathrm{Nm}_{L/K}((y)) = (\mathrm{Nm}_{L/K}(y))$.

Proposition 6.6.3: For any two fractional ideals $\mathfrak{b}, \mathfrak{b}'$ of B we have

$$\mathrm{Nm}_{L/K}(\mathfrak{b}\mathfrak{b}') \;=\; \mathrm{Nm}_{L/K}(\mathfrak{b}) \cdot \mathrm{Nm}_{L/K}(\mathfrak{b}').$$

Proposition 6.6.4: For any fractional ideal $\mathfrak{c}$ of C we have

$$\mathrm{Nm}_{L/K}(\mathrm{Nm}_{M/L}(\mathfrak{c})) \;=\; \mathrm{Nm}_{M/K}(\mathfrak{c}).$$

Proposition 6.6.5: For any fractional ideal $\mathfrak{a}$ of A we have $\mathrm{Nm}_{L/K}(\mathfrak{a}B) = \mathfrak{a}^n$.

Proposition 6.6.6: For any prime $\mathfrak{q} \subset B$ above $\mathfrak{p} \subset A$ we have $\mathrm{Nm}_{L/K}(\mathfrak{q}) = \mathfrak{p}^{e_{\mathfrak{q}|\mathfrak{p}}}$.

6.7 Different

Recall from Proposition 1.7.1 that we have the non-degenerate symmetric K-bilinear form

$$L \times L \longrightarrow K, \quad (x, y) \mapsto \mathrm{Tr}_{L/K}(xy).$$

Proposition 6.7.1: The subset

$$\mathfrak{d} \;:=\; \big\{x \in L \;\big|\; \forall y \in B\colon\ \mathrm{Tr}_{L/K}(xy) \in A\big\}$$

is a fractional ideal of B which contains B.

Definition 6.7.2: The ideal $\mathrm{diff}_{B/A} := \mathfrak{d}^{-1} \subset B$ is called the *different of B over A*.

Proposition 6.7.3: Suppose that $B = A[\beta]$ and let $f \in A[X]$ be the minimal polynomial of β above K. Then $\mathrm{diff}_{B/A} = \big(\frac{df}{dX}(\beta)\big)$.

Proposition 6.7.4: In general $\mathrm{diff}_{B/A}$ is the ideal that is generated by $\frac{df}{dX}(\beta)$ for all $\beta \in B$ with minimal polynomial f over K.

Proposition 6.7.5: We have $\mathrm{diff}_{C/A} = \mathrm{diff}_{C/B} \cdot \mathrm{diff}_{B/A}$.

Theorem 6.7.6: For any prime $\mathfrak{q}$ of B above a prime $\mathfrak{p}$ of A we have $\mathfrak{q} \nmid \mathrm{diff}_{B/A}$ if and only if $\mathfrak{q}$ is unramified over $\mathfrak{p}$.

6.8 Relative discriminant

Definition 6.8.1 The *relative discriminant of B/A* is the ideal of A that is generated by the discriminants

$$\mathrm{disc}(b_1, \ldots, b_n) \;=\; \det\big(\mathrm{Tr}_{L/K}(b_i b_j)\big)_{i,j=1,\ldots,n}$$

for all tuples $(b_1, \ldots, b_n)$ in B.

Proposition 6.8.2: We have $\mathrm{disc}_{B/A} = \mathrm{Nm}_{L/K}(\mathrm{diff}_{B/A})$.

Proposition 6.8.3: We have $\mathrm{disc}_{C/A} = \mathrm{Nm}_{L/K}(\mathrm{disc}_{C/B}) \cdot \mathrm{disc}_{B/A}^{[M/L]}$.

Theorem 6.8.4: (a) A prime $\mathfrak{p} \subset A$ is ramified in B if and only if $\mathfrak{p} | \operatorname{disc}_{B/A}$.

(b) At most finitely many primes of A are ramified in B.

Theorem 6.8.5: For any number field $K \neq \mathbb{Q}$ there exists a rational prime which is ramified in $\mathcal{O}_K$.

Example 6.8.6: Consider distinct primes $p_1 \equiv \ldots \equiv p_r \equiv 1 \bmod (4)$ with $r \geqslant 1$. Then the extension $\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_r}\,)/\mathbb{Q}(\sqrt{p_1 \cdots p_r}\,)$ is everywhere unramified.

7 Zeta functions

7.1 Riemann zeta function

Definition 7.1.1: The *Riemann zeta function* is defined by the series

$$\zeta(s) \ := \ \sum_{n=1}^{\infty} n^{-s}.$$

Proposition 7.1.2: This series converges absolutely and locally uniformly for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a holomorphic function there.

Lemma 7.1.3: For all $\mathrm{Re}(s) > 1$ we have

$$\zeta(s) \ = \ \frac{s}{s-1} - s \cdot \int_1^{\infty} (x - \lfloor x \rfloor) x^{-s-1}\, dx.$$

Proposition 7.1.4: The function $\zeta(s) - \frac{1}{s-1}$ extends uniquely to a holomorphic function on the region $\mathrm{Re}(s) > 0$.

Remark 7.1.5: We may see later that $\zeta(s)$ extends uniquely to a meromorphic function on $\mathbb{C}$ with a single pole at $s = 1$. This extension is again denoted by $\zeta(s)$.

Throughout the following we use the branch of the logarithm with $\log 1 = 0$.

Proposition 7.1.6: An infinite product of non-zero complex numbers $\prod_{k \geqslant 1} z_k$ converges to a non-zero value if and only if $\lim_{k\to\infty} z_k = 1$ and $\sum_{k \geqslant 1} \log z_k$ converges.

Proposition 7.1.7: For all $\mathrm{Re}(s) > 1$ we have the *Euler product*

$$\zeta(s) \ = \ \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \ \neq \ 0.$$

Proposition 7.1.8: We have

$$\sum_{p \text{ prime}} p^{-s} \ = \ \log \tfrac{1}{s-1} + O(1) \ \text{ for real } s \to 1+.$$

Definition 7.1.9: For $x \in \mathbb{R}$ we denote the number of primes $\leqslant x$ by $\pi(x)$.

Corollary 7.1.10: There is no $\varepsilon > 0$ such that for $x \to \infty$ we have

$$\pi(x) \ = \ O\Big(\frac{x}{(\log x)^{1+\varepsilon}}\Big).$$

In particular there exist infinitely many primes.

7.2 Dedekind zeta function

Fix a number field K of degree n over $\mathbb{Q}$.

Definition 7.2.1: The *Dedekind zeta function of* K is defined by the series

$$\zeta_K(s) \;:=\; \sum_{\mathfrak{a}} \mathrm{Nm}(\mathfrak{a})^{-s},$$

where the sum extends over all non-zero ideals $\mathfrak{a} \subset \mathcal{O}_K$.

Proposition 7.2.2: This series converges absolutely and locally uniformly for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a holomorphic function there, and we have the *Euler product*

$$\zeta_K(s) \;=\; \prod_{\mathfrak{p}} \bigl(1 - \mathrm{Nm}(\mathfrak{p})^{-s}\bigr)^{-1} \;\neq\; 0,$$

extended over all maximal ideals $\mathfrak{p} \subset \mathcal{O}_K$.

Proposition 7.2.3: We have

$$\log \zeta_K(s) \;=\; \sum_{\mathfrak{p}} \mathrm{Nm}(\mathfrak{p})^{-s} + \bigl(\text{holomorphic for } \mathrm{Re}(s) > \tfrac{1}{2}\bigr).$$

Theorem 7.2.4: The function $\zeta_K(s)$ extends uniquely to a meromorphic function on the region $\mathrm{Re}(s) > 1 - \frac{1}{n}$ which is holomorphic except for a pole of order 1 at $s = 1$.

Proposition 7.2.5: We have

$$\sum_{\mathfrak{p}} \mathrm{Nm}(\mathfrak{p})^{-s} \;=\; \log \tfrac{1}{s-1} + O(1) \;\text{ for real } s \to 1+.$$

Corollary 7.2.6: There exist infinitely many rational primes that split totally in $\mathcal{O}_K$.

7.3 Analytic class number formula

As before we set $\Sigma := \mathrm{Hom}(K, \mathbb{C})$ and let r be the number of embeddings $K \hookrightarrow \mathbb{R}$ and s the number of pairs of complex conjugate non-real embeddings $K \hookrightarrow \mathbb{C}$. With $K_{\mathbb{C}} := \mathbb{C}^{\Sigma}$ and

$$K_{\mathbb{R}} \;:=\; \bigl\{(z_\sigma)_\sigma \in K_{\mathbb{C}} \bigm| \forall \sigma \in \Sigma\colon z_{\bar{\sigma}} = \bar{z}_\sigma \bigr\}$$

as in §3.4 we then have

$$K_{\mathbb{R}} \cap \mathbb{R}^{\Sigma} \;=\; \bigl\{(t_\sigma)_\sigma \in \mathbb{R}^{\Sigma} \bigm| \forall \sigma \in \Sigma\colon t_{\bar{\sigma}} = t_\sigma \bigr\}.$$

The $\mathbb{R}$-subspace

$$H := \ker\bigl(\mathrm{Tr}\colon K_{\mathbb{R}} \cap \mathbb{R}^{\Sigma} \to \mathbb{R}\bigr)$$

from §5.2 therefore becomes a euclidean vector space by its embedding $H \subset K_{\mathbb{R}} \subset K_{\mathbb{C}}$ and the scalar product from §4.1. By §2.2 it is thus endowed with a canonical translation invariant measure $d\,\mathrm{vol}$. Recall from Theorem 5.3.1 that $\Gamma := \ell(j(\mathcal{O}_K^\times))$ is a complete lattice in H.

Definition 7.3.1: The *regulator of* K is the real number

$$R \;:=\; \mathrm{vol}(H/\Gamma) \;>\; 0.$$

Let $w := |\mu(K)|$ denote the number of roots of unity in K and let $h := |\,\mathrm{Cl}(\mathcal{O}_K)|$ the class number.

Theorem 7.2.7: *Analytic class number formula:* The residue of $\zeta_K(s)$ at $s = 1$ is

$$\mathrm{Res}_{s=1}\,\zeta_K(s) \;=\; \frac{2^r(2\pi)^s Rh}{w\sqrt{|d_K|}} \;>\; 0.$$

7.4 Dirichlet density

Consider a number field K and a subset A of the set P of maximal ideals of $\mathcal{O}_K$.

Definition 7.4.1: (a) The value

$$\overline{\mu}(A) \;:=\; \limsup_{s\to1+} \frac{\sum_{\mathfrak{p}\in A}\mathrm{Nm}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in P}\mathrm{Nm}(\mathfrak{p})^{-s}}$$

is called the *upper Dirichlet density of* A.

(b) The value

$$\underline{\mu}(A) \;:=\; \liminf_{s\to1+} \frac{\sum_{\mathfrak{p}\in A}\mathrm{Nm}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in P}\mathrm{Nm}(\mathfrak{p})^{-s}}$$

is called the *lower Dirichlet density of* A.

(c) If these coincide, their common value

$$\mu(A) \;:=\; \lim_{s\to1+} \frac{\sum_{\mathfrak{p}\in A}\mathrm{Nm}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in P}\mathrm{Nm}(\mathfrak{p})^{-s}}$$

is called the *Dirichlet density of* A.

Proposition 7.4.2: (a) We have $0 \leqslant \underline{\mu}(A) \leqslant \overline{\mu}(A) \leqslant 1$.

(b) For any subset $B \subset A$ we have $\overline{\mu}(B) \leqslant \overline{\mu}(A)$ and $\underline{\mu}(B) \leqslant \underline{\mu}(A)$, and also $\mu(B) \leqslant \mu(A)$ if these exist.

(c) We have $\mu(A) = 0$ if A is finite.

(d) We have $\mu(A) = 1$ if $P \smallsetminus A$ is finite.

(e) For any disjoint subsets $A, B \subset P$, if two of $\mu(A)$, $\mu(B)$, $\mu(A \cup B)$ exist, then so does the third and we have $\mu(A) + \mu(B) = \mu(A \cup B)$.

Proposition-Definition 7.4.3: If the *natural density of A*

$$\gamma(A) \;:=\; \lim_{x\to\infty} \frac{\big|\{\mathfrak{p}\in A \mid \mathrm{Nm}(\mathfrak{p})\leqslant x\}\big|}{\big|\{\mathfrak{p}\in P \mid \mathrm{Nm}(\mathfrak{p})\leqslant x\}\big|}$$

exists, so does the Dirichlet density $\mu(A)$ and they are equal.

7.5 Primes of absolute degree 1

Definition 7.5.1: The *absolute degree* of a prime $\mathfrak{p}$ of $\mathcal{O}_K$ is the degree of $k(\mathfrak{p})$ over its prime field.

Proposition 7.5.2: The set of primes of absolute degree 1 has Dirichlet density 1.

Proposition 7.5.3: A subset $A\subset P$ has a Dirichlet density if and only if the set of all $\mathfrak{p}\in A$ of absolute degree 1 has a Dirichlet density, and then they are equal.

For any finite galois extension of number fields L/K we let $\mathrm{Split}_{L/K}$ denote the set of primes $\mathfrak{p}\subset\mathcal{O}_K$ that are totally split in $\mathcal{O}_L$.

Proposition 7.5.4: $\mathrm{Split}_{L/K}$ has Dirichlet density $\frac{1}{[L/K]}$. In particular it is infinite.

Now consider two finite galois extensions of number fields $L,L'/K$.

Proposition 7.5.5: Then $\mathrm{Split}_{LL'/K}=\mathrm{Split}_{L/K}\cap\mathrm{Split}_{L'/K}$.

Proposition 7.5.6: The following are equivalent:

(a) $L\subset L'$.

(b) $\mathrm{Split}_{L'/K}\subset\mathrm{Split}_{L/K}$.

(c) $\mu(\mathrm{Split}_{L'/K}\smallsetminus\mathrm{Split}_{L/K})<\frac{1}{2[L/K]}$.

Proposition 7.5.7: The following are equivalent:

(a) $L=L'$.

(b) $\mathrm{Split}_{L'/K}$ and $\mathrm{Split}_{L/K}$ differ only by a set of Dirichlet density 0.

In particular, a number field K that is galois over $\mathbb{Q}$ is uniquely determined by the set of rational primes p that split totally in K.

7.6 Dirichlet L-series

Definition 7.6.1: (a) A homomorphism $\chi\colon(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$ is called a *Dirichlet character of modulus $N\geqslant 1$*.

(b) The *conductor* of such χ is the smallest divisor $N'|N$ such that χ factors through a homomorphism $(\mathbb{Z}/N'\mathbb{Z})^\times\to\mathbb{C}^\times$.

(c) Such χ is called *primitive* if $N' = N$.

(d) Such χ is called *principal* if $N' = 1$, that is, if χ is the trivial homomorphism.

Convention 7.6.2: One often identifies a Dirichlet character χ of modulus N with a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ by setting

$$\chi(a) \ := \ \begin{cases} \chi(a \bmod (N)) & \text{if } \gcd(a,N)=1, \\ 0 & \text{otherwise.} \end{cases}$$

Caution 7.6.3: When the conductor N' is smaller than the modulus N, one has to be somewhat careful with the divisors of N/N'.

Definition 7.6.4: The *Dirichlet L-function* associated to any Dirichlet character χ is

$$L(\chi,s) \ := \ \sum_{n\geqslant 1} \chi(n)n^{-s}.$$

Proposition 7.6.5: This series converges absolutely and locally uniformly for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a holomorphic function there.

Proposition 7.6.6: For all $\mathrm{Re}(s) > 1$ we have the *Euler product*

$$L(\chi,s) \ = \ \prod_{p\nmid N} (1-\chi(p)p^{-s})^{-1}.$$

Proposition 7.6.7: If a Dirichlet character χ of modulus N corresponds to a primitive Dirichlet character χ' of modulus N', then

$$L(\chi',s) \ = \ L(\chi,s) \cdot \prod_{p\mid N,\, p\nmid N'} (1-p^{-s})^{-1}.$$

Proposition 7.6.8: (a) For the principal Dirichlet character χ of modulus 1 we have $L(\chi,s) = \zeta(s)$.

(b) For every non-principal Dirichlet character χ the function $L(\chi,s)$ extends uniquely to a holomorphic function on the region $\mathrm{Re}(s) > 0$.

Theorem 7.6.9: The zeta function $\zeta_K(s)$ of the field $K := \mathbb{Q}(\mu_N)$ is the product of the L-functions $L(\chi,s)$ for all primitive Dirichlet characters χ of conductor dividing N.

Theorem 7.6.10: For any non-principal Dirichlet character χ we have $L(\chi,1) \neq 0$.

Proposition 7.6.11: For any non-principal Dirichlet character χ we have

$$\sum_{p \text{ prime}} \chi(p)p^{-s} \ = \ O(1) \quad \text{for real } s \to 1+.$$

7.7 Primes in arithmetic progressions

Theorem 7.7.1: For any coprime integers a and $N \geqslant 1$ the set of rational primes $p \equiv a \bmod (N)$ has Dirichlet density $\frac{1}{\varphi(N)}$. In particular it is infinite.

This can also be viewed as the special case $L = \mathbb{Q}(\mu_N)$ and $K = \mathbb{Q}$ of the following general theorem:

Theorem 7.7.2: *Cebotarev density theorem:* Let L/K be a Galois extension of number fields with Galois group Γ. For any $\gamma \in \Gamma$ consider its conjugacy class $O_\Gamma(\gamma) := \{{}^{\gamma'}\gamma \mid \gamma' \in \Gamma\}$. Then the set of primes $\mathfrak{p} \subset \mathcal{O}_K$ that are unramified in $\mathcal{O}_L$ and whose Frobenius substitution lies in $O_\Gamma(\gamma)$ has the Dirichlet density $\frac{|O_\Gamma(\gamma)|}{|\Gamma|}$.

References

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09. 11. 2023: Assumption 6.2.1 added and the rest of §6.2 renumbered. As a result of Assumption 6.2.1 substantial changes in §6.3–4 and reformulations in 6.7.6 and 6.8.4–7.

08. 11. 2023: Corrected Proposition 5.4.5: $\varepsilon \geqslant \frac{\sqrt{D}+\sqrt{D-4}}{2} > 1$.

07. 11. 2023: Chapter 7 added.

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25. 10. 2023: Corrected $\sqrt{|\operatorname{disc}(\mathfrak{a})|}$ in Proposition 4.2.1.

20. 10. 2023: Corrected Theorem 4.2.2.

18. 10. 2023: Corrected Definition 3.6.4 and two typos in Proposition 4.1.2.

13. 10. 2023: Proposition 3.2.1 and typos in §3.6 corrected.

12. 10. 2023: Theorem 3.6.7 expanded.

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