Exercise sheet 2

1. Show that if $A_1 \subset G_1$ and $A_2 \subset G_2$ are Sidon sets with $|A_i| \geqslant 2$, then $A_1 \times A_2$ is not a Sidon set in $G_1 \times G_2$.

2. Let G be a finite abelian group. Let $\alpha \in G$ be a fixed element. A subset $A \subset G$ is called a *symmetric Sidon set* with center α if $A = \alpha - A$ (i.e., for any x in A, the element $\alpha - x$ is also in A) and if the equation
$$a + b = c + d$$
with $(a, b, c, d) \in A^4$ implies that $a \in \{c, d\}$ or $a + b = \alpha$.

   (a) Let E be a field with characteristic different from 3. Prove that the set
   $$A = \{(x, x^3) \mid x \in E\} \subset E \times E$$
   is a symmetric Sidon set with center 0.

   (b) Prove that if $A \subset G$ is a symmetric Sidon set, then it contains a subset $A' \subset A$ with $|A'| \geqslant (|A| - 1)/2$ such that A' is a Sidon set.

   (c) Let G be a finite abelian group and $A \subset \widehat{G}$ a finite set of characters of G. If A is a symmetric Sidon set with center α, prove that
   $$\sum_{x\in G}\Big|\sum_{\chi\in A}\lambda_\chi\chi(x)\Big|^4 \leqslant 3\Big(\sum_{\chi\in A}|\lambda_\chi|^2\Big)^2.$$

3. Let G be an abelian group, denoted additively. For a finite subset $A \subset G$, we denote by $E(A)$ the number of quadruples $(a, b, c, d) \in A^4$ such that $a + b = c + d$.

   (a) Show that A is a Sidon set in G if and only if $E(A) = 2|A|^2 - |A|$.

   The remainder of the exercise shows that a finite set A may satisfy $E(A) = 2|A|^2 + O(|A|)$, but not contain any Sidon subset of size $\sim |A|$. We take $G = \mathbf{Z}$.

   (b) Show that for all large integers N, there exists a Sidon set $A \subset \{1, \ldots, N\} \cap 2\mathbf{Z}$ with $|A| \to +\infty$ as $N \to +\infty$.

   (c) Consider a Sidon set $A \subset \{1, \ldots, N\} \cap 2\mathbf{Z}$. Define
   $$A' = A \cup \{a + N2^{a+1} \mid a \in A\} \cup \{a - N2^{a+1} \mid a \in A\} \subset \mathbf{Z}.$$

(d) Show that if $A'' \subset A'$ is a Sidon set, we have $|A''| \leqslant \frac{2}{3}|A'|$.

(e) Let

$$x_1 + x_2 = x_3 + x_4,$$

with

$$x_i = a_i + \varepsilon_i N 2^{a_i+1}, \quad a_i \in A, \quad \varepsilon_i \in \{-1, 0, 1\},$$

Show that $a_1 + a_2 = a_3 + a_4$.

(f) Suppose that $a_1 = a_3$, hence $a_2 = a_4$. Show that

$$(\varepsilon_1 - \varepsilon_3)2^{a_1} = (\varepsilon_4 - \varepsilon_2)2^{a_2}.$$

(g) Deduce that $x_1 = x_3$ if $\varepsilon_1 = \varepsilon_3$ or $\varepsilon_2 = \varepsilon_4$.

(h) Suppose further that $\varepsilon_1 \neq \varepsilon_3$ and $\varepsilon_2 \neq \varepsilon_4$. Show that $a_1 = a_2 = a_3 = a_4$ and $\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4$.

(i) Conclude that if $x_1 \notin \{x_3, x_4\}$, then (x_1, x_2, x_3, x_4) has one of the forms

$$\begin{aligned} &(a + N2^{a+1}, a - N2^{a+1}, a, a), \qquad &&(a - N2^{a+1}, a + N2^{a+1}, a, a),\\ &(a, a, a - N2^{a+1}, a + N2^{a+1}), \qquad &&(a, a, a + N2^{a+1}, a - N2^{a+1}), \end{aligned}$$

for some $a \in A$. (Hint: consider the various possibilities for $(\varepsilon_1, \ldots, \varepsilon_4)$ for given $(\varepsilon_3, \varepsilon_4)$.

(j) Deduce that

$$E(A'') = 2|A''|^2 + O(|A''|).$$