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Exercise sheet 3

1. For any integer $N \geqslant 1$, find examples of sets A and B of positive integers such that $|A| = |B| = N$ and
$$\frac{|2A|}{A} \leqslant 2, \qquad \frac{|2B|}{|B|} \leqslant 2,$$
but
$$\frac{|2(A \cup B)|}{|A \cup B|} \geqslant \frac{N}{2}.$$

2. For any integer $N \geqslant 1$, find examples of sets A and B of positive integers such that $|A| = |B| = N$ and
$$\frac{|2A|}{A} \leqslant 10, \qquad \frac{|2B|}{|B|} \leqslant 10,$$
but
$$\frac{|2(A \cap B)|}{|A \cap B|} \geqslant \frac{N^{1/2}}{10}.$$

3. Let A_1, A_2, A_3 be non-empty finite subsets of some group G. If $\alpha \geqslant 1$ is such that
$$|A_j \cap A_3| \geqslant \frac{|A_j|}{\alpha}, \qquad |A_i \cdot A_j| \leqslant \alpha |A_j|$$
for all i and j, then show that
$$|A_1 \cdot A_2| \leqslant \alpha^6 |A_3|.$$
(Hint: use the Ruzsa triangle inequality suitably)

4. Let G be a finite *abelian* group and A, B non-empty subsets of G. Let
$$r(x) = \sum_{\substack{(a,b) \in A \times B \\ a+b=x}} 1$$
be the representation function for $A + B$.

   (a) Show that $r(x) = |A \cap (x - B)|$.

(b) Show that

$$E(A,B) = \sum_{x \in (A-A)\cap(B-B)} |A \cap (x+A)|\,|B \cap (x+B)|.$$

5. Let G be a finite group and A, B non-empty subsets of G. Let $x_0 \in A \cdot B$.

   (a) Prove that

   $$|\{(a,b) \in A \times B \mid ab = x_0\} \times (B \cdot A)| \leqslant |B \cdot A^{-1}|\,|B^{-1} \cdot A|.$$

   (Hint: construct an injective map from the left-hand set to the cartesian product $B \cdot A^{-1} \times B^{-1} \cdot A$.)

   (b) Deduce that if $x \in A \cdot B$, then

   $$|A \cap xB^{-1}| \leqslant \frac{|B \cdot A^{-1}||B^{-1} \cdot A|}{|A \cdot B|},$$

   (c) If G is abelian, deduce that

   $$|A \cap xB^{-1}| \leqslant \frac{|B \cdot A^{-1}|^2}{|A \cdot B|}.$$

6. Let G be a finite abelian group.

   (a) If H_1 and H_2 are subgroups of G, then show that the Ruzsa distance $d(H_1, H_2)$ satisfies

   $$d(H_1, H_2) = \log\Big(\frac{\sqrt{|H_1||H_2|}}{|H_1 \cap H_2|}\Big).$$

   (b) Show that

   $$d(H_1,H_2) = d(H_1, H_1{+}H_2) + d(H_1{+}H_2, H_2) = d(H_1, H_1{\cap}H_2) + d(H_1{\cap}H_2, H_2).$$

7. Let G be a finite abelian group and $A \subset G$ a non-empty subset such that

   $$|2A - 2A| < 2|A|.$$

   (a) Show that there exists $x_0 \in G$ such that

   $$A - 2A \subset A - A + x_0.$$

   (b) Deduce that $A - A$ is a subgroup of G.