Exercise sheet 3

1. For any integer $N \geqslant 1$, find examples of sets A and B of positive integers such that $|A| = |B| = N$ and
$$\frac{|2A|}{|A|} \leqslant 2, \qquad \frac{|2B|}{|B|} \leqslant 2,$$
but
$$\frac{|2(A \cup B)|}{|A \cup B|} \geqslant \frac{N}{2}.$$
Solution. Let $A = \{1, \dots, N\}$ and $B = \{N^2 + N, N^2 + 2N, \dots, N^2 + N \cdot N\}$. It is clear that
$$\frac{|2A|}{|A|} \leqslant 2, \qquad \frac{|2B|}{|B|} \leqslant 2.$$
We observe that, $A + B \subset 2(A \cup B)$ and
$$A + B = \{N^2 + aN + b : a, b \in [N]\},$$
so $|A + B| = N^2$. On the other hand $|A \cup B| = 2N$ thus
$$\frac{|2(A \cup B)|}{|A \cup B|} \geqslant \frac{N^2}{2N} = \frac{N}{2}.$$

2. For any integer $N \geqslant 1$, find examples of sets A and B of positive integers such that $|A| = |B| = N$ and
$$\frac{|2A|}{|A|} \leqslant 10, \qquad \frac{|2B|}{|B|} \leqslant 10,$$
but
$$\frac{|2(A \cap B)|}{|A \cap B|} \geqslant \frac{N^{1/2}}{10}.$$
Solution.

From Theorem 2.3.13 (1), for any interval of $\mathbb{Z}$ of length N we can find a Sidon set of size $\geqslant N^{1/2}/2$. Let S be a Sidon set of size $\geqslant N^{1/2}/4$ in $\{1, \cdots N/2\}$. We let
$$A = S \cup \{N/2, \cdots, N\}$$
$$B = S \cup \{N, \cdots, 3N/2\}.$$

We observe that $|2A| \leqslant |2A'| + |2S| + |A' + S| \leqslant 3N$. On the other hand $|A| \geqslant \frac{N^{1/2}}{4} + \frac{N}{2}$, thus it holds that

$$\frac{|2A|}{|A|} \leqslant 10.$$

A similar argument works to show that

$$\frac{|2B|}{|B|} \leqslant 10.$$

We observe that $A \cap B = S$, thus $|2(A \cap B)| \geqslant \frac{N}{16}$ and we can conclude the result.

3. Let A_1, A_2, A_3 be non-empty finite subsets of some group G. If $\alpha \geqslant 1$ is such that

$$|A_j \cap A_3| \geqslant \frac{|A_j|}{\alpha}, \qquad |A_j \cdot A_j| \leqslant \alpha |A_j|$$

for $1 \leqslant j \leqslant 3$, then show that

$$|A_1 \cdot A_2| \leqslant \alpha^6 |A_3|.$$

(Hint: use the Ruzsa triangle inequality suitably)

Solution.

We recall that Rusza inequality can be framed as

$$|A \cdot C^{-1}| \leqslant \frac{|A \cdot B^{-1}| \cdot |B \cdot C^{-1}|}{|B|}.$$

We apply this inequality twice:

$$\begin{aligned} |A_1 \cdot A_2| &\leqslant \frac{|A_1 \cdot (A_1 \cap A_3)| \cdot |(A_1 \cap A_3)^{-1} \cdot A_2|}{|A_1 \cap A_3|} \\ &\leqslant \frac{|A_1 \cdot (A_1 \cap A_3)| \cdot |(A_1 \cap A_3)^{-1} \cdot (A_2 \cap A_3)^{-1}||(A_2 \cap A_3) \cdot A_2|}{|A_1 \cap A_3||A_2 \cap A_3|}. \end{aligned}$$

Thus, using the hyphotesis we get

$$\begin{aligned} |A_1 \cdot A_2| &\frac{\alpha|A_1|\alpha|A_3|\alpha|A_2|}{\frac{1}{\alpha}|A_1|\frac{1}{\alpha}|A_2|} \\ &\leqslant \alpha^5 |A_3|. \end{aligned}$$

4. Let G be a finite *abelian* group and A, B non-empty subsets of G. Let

$$r(x) = \sum_{\substack{(a,b) \in A \times B \\ a+b=x}} 1$$

be the representation function for $A + B$.

(a) Show that $r(x) = |A \cap (x - B)|$.

(b) Show that

$$E(A, B) = \sum_{x \in (A-A) \cap (B-B)} |A \cap (x + A)|\,|B \cap (x + B)|.$$

Solution.

(a) Observe that $(a, b) \in A \times B$ is such that $a + b = x$ if and only if $a = x - b \in (x - B) \cap A$. Thus $r(x) = |A \cap (x - B)|$.

(b)

$$\begin{aligned}
E(A, B) &= \sum_{x \in A \times B} |\{(a, b) \in A \times B : a + b = x\}| \\
&= \sum_{x \in A+B} |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2 = x\}| \\
&= |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2\}| \\
&= |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - a_2 = b_2 - b_1\}| \\
&= \sum_{z \in (A-A) \cap (B-B)} |\{(a_1, a_2) \in A \times A : a_1 - a_2 = z\}|\,|\{(b_1, b_2) \in B \times B : b_2 - b_1 = z\}| \\
&= \sum_{z \in (A-A) \cap (B-B)} |A \cap (z + A)||B \cap (z + B)|
\end{aligned}$$

where the last equality holds from a similar argument to the solution of the first item.

5. Let G be a finite group and A, B non-empty subsets of G. Let $x_0 \in A \cdot B$.

(a) Prove that

$$|\{(a, b) \in A \times B \mid ab = x_0\} \times (B \cdot A)| \leqslant |B \cdot A^{-1}|\,|B^{-1} \cdot A|.$$

(Hint: construct an injective map from the left-hand set to the cartesian product $B \cdot A^{-1} \times B^{-1} \cdot A$.)

(b) Deduce that if $x \in A \cdot B$, then

$$|A \cap xB^{-1}| \leqslant \frac{|B \cdot A^{-1}||B^{-1} \cdot A|}{|A \cdot B|},$$

(c) If G is abelian, deduce that

$$|A \cap xB^{-1}| \leqslant \frac{|B \cdot A^{-1}|^2}{|B \cdot A|}.$$

Solution.

(a) For $x \in B \cdot A$, chose $a(x) \in A$ and $b(x) \in B$ such that $b(x)a(x) = x$. Define the map $f : \{(a, b) \in A \times B | ab = x_0\} \times B \cdot A \to B \cdot A^{-1} \times B^{-1} \times B^{-1} \cdot A$, $f(a, b, x) = (b(x)a^{-1}, b^{-1}a(x))$. Observe that if $f(a, b, x) = (u, v)$ then $u \cdot x_0 \cdot v = x$, so we can recover $a(x)$ and $b(x)$ and consequently a and b and we conclude that f is injective.

(b) We observe that $(a, b) \in A \times B$ satisfies $ab = x$ if an only if $a = xb^{-1} \in A \cap xB^{-1}$. Thus, from the previous item we deduce that

$$|A \cap xB^{-1}| = |\{(a, b) \in A \times B \mid ab = x\}| \leqslant \frac{|B \cdot A^{-1}|\,|B^{-1} \cdot A|}{|(B \cdot A)|}.$$

(c) If G is abelian then $|B^{-1} \cdot A| = |(B^{-1} \cdot A)^{-1}| = |A^{-1} \cdot B| = |B \cdot A^{-1}|$ and $|A \cdot B| = |B \cdot A|$. The inequality follow from the previous item and these observations.

6. Let G be a finite abelian group.

(a) If H_1 and H_2 are subgroups of G, then show that the Ruzsa distance $d(H_1, H_2)$ satisfies

$$d(H_1, H_2) = \log\Big(\frac{\sqrt{|H_1||H_2|}}{|H_1 \cap H_2|}\Big).$$

(b) Show that

$$d(H_1, H_2) = d(H_1, H_1+H_2)+d(H_1+H_2, H_2) = d(H_1, H_1\cap H_2)+d(H_1\cap H_2, H_2).$$

Solution.

(a) We observe that it suffices to prove that

$$|H_1 + H_2^{-1}| \cdot |H_1 \cap H_2| = |H_1| \cdot |H_2|.$$

For every $x \in H_1 \cdot -H_2$ we choose $h_1(x) \in H_1$ and $h_2(x) \in H_2$ such that $h_1(x) - h_2(x) = x$.
We define the function $f : H_1 - H_2 \times H_1 \cap H_2 \to H_1 \times H_2$, $f(x, h) = (h_1(x) + h, -h + h_2(x))$ and prove that it is a bijection. First, observe that the function is well defined because H_1 and H_2 are subgroups of G.
Now, bserve that it is injective. Indeed, for $(u, v) \in f(H_1 - H_2 \times H_1 \cap H_2)$ we have that $u \cdot v = x$, so we can recover $h_1(x)$ and $h_2(x)$. Thus, $h = -h_1(x) + u$.
To prove that it is surjective, let $(h_1, h_2) \in H_1 + H_2$ be arbitrary. Let $x = h_1 - h_2$. Since $h_1 + h_2 = h_1(x) + h_2(x)$, we let $h = h_1 - h_1(x) = h_2(x) - h_2 \in H_1 \cap H_2$ and observe that

$$f(x, h) = (h_1(x) + h_1 - h_1(x), -h_2(x) + h_2 + h_2(x)) = (h_1, h_2).$$

(b) To prove the first equality observe that

$$\exp(d(H_1, H_1 + H_2) + d(H_2, H_1 + H_2)) = \frac{|H_1 + H_1 + H_2||H_2 + H_1 + H_2|}{\sqrt{|H_1||H_1 + H_2|^2|H_2|}},$$

and since H_1 and H_2 are subgroups of G it holds that

$$\exp(d(H_1, H_1 + H_2) + d(H_2, H_1 + H_2)) = \frac{|H_1 + H_2|}{\sqrt{|H_1||H_2|}} = \exp(H_1, H_2).$$

To prove the second equality we use item a). Observe that $H_1 \cap H_2$ is a subgroup of G so we can write

$$\begin{aligned}\exp(d(H_1, H_1 \cap H_2) + d(H_2, H_1 \cap H_2)) &= \frac{\sqrt{|H_1||H_1 \cap H_2|^2|H_2|}}{|H_1 \cap H_2|^2} \\ &== \frac{\sqrt{|H_1||H_2|}}{|H_1 \cap H_2|} = \exp(d(H_1, H_2)).\end{aligned}$$

7. Let G be a finite abelian group and $A \subset G$ a non-empty subset such that

$$|2A - 2A| < 2|A|.$$

(a) Show that there exists $x_0 \in G$ such that

$$A - 2A \subset A - A + x_0.$$

(b) Deduce that $A - A$ is a subgroup of G.

Solution.

(a) Observe that we have $|A + (A - 2A)| < 2|A|$. From Ruzsa's covering lemma we can find $X \subset G$ such that

$$A - 2A \subset A - A + X,$$

and $|X| \leqslant \frac{|A+(A-2A)|}{|A|} < 2$, so $X = \{x_0\}$.

(b) First we observe that $0 \in A - A$ and if $x = a_1 - a_2 \in A - A$ then $-x = a_2 - a_1 \in A - A$. Thus, to prove that $A - A$ is a subgroup of G, it suffices to prove that $2(A - A) \subset A - A$.

From the previous item we can conclude that

$$2A - A \subset A - A - x_0,$$

thus,

$$2A - 2A = A + (A - 2A) \subset A + (A - A + x_0) \subset A - A + x_0 - x_0 = A = A.$$