Exercise sheet 5

1. Let G be a finite abelian group. For any subsets A and B of G, we denote

$$r_{A,-B}(x) = |\{(a,b) \in A \times B \mid a - b = x\}|.$$

(a) Show that for any sets A and B, we have

$$\sum_{x \in G} r_{A,-B}(x)^2 = \sum_{x \in G} r_{A,-A}(x) r_{B,-B}(x).$$

(b) We assume from now on that A is a Sidon set in G. Prove that

$$\sum_{x \in G} r_{A,-A}(x) r_{B,-B}(x) \leqslant |A||B| + |B|^2 - |B|.$$

(c) Deduce from the previous questions that

$$\sum_{x \in G} \Big(r_{A,-B}(x) - \frac{|A||B|}{|G|} \Big)^2 \leqslant |B|(|A| - 1) + \frac{|B|^2(|G| - |A|^2)}{|G|}.$$

(d) Let also C be a subset of G and define

$$N = |\{(b,c) \in B \times C \mid b + c \in A\}|.$$

Show that

$$N - \frac{|A||B||C|}{|G|} = \sum_{c \in C} \Big(r_{A,-B}(c) - \frac{|A||B|}{|G|} \Big).$$

(e) Deduce that

$$N - \frac{|A||B||C|}{|G|} \leqslant |C|^{1/2} \Big(|B|(|A| - 1) + \frac{|B|^2(|G| - |A|^2}{|G|} \Big)^{1/2}.$$

(f) Define δ by $|A| = |G|^{\frac{1}{2}} - \delta$. Show that

$$N = \frac{|A||B||C|}{|G|} + \theta(|B||C|\sqrt{|G|})^{1/2},$$

where

$$\theta \leqslant 1 + \frac{|B|}{|G|} \max(0, \delta), \qquad \theta \leqslant 1 + \frac{|C|}{|G|} \max(0, \delta).$$

(g) Show that

$$|C|\,|A\cap B| \leqslant |\{(x,y)\in -C\times(B+C) \mid x+y\in A\}|.$$

(h) Deduce that

$$|A\cap B| \leqslant \frac{|B+C||A|}{|G|} + \theta\Big(\frac{|B+C|}{|C|}\Big)^{1/2}|G|^{1/4}.$$

2. Let p be a prime number. Let $P\subset \mathbf{F}_p^2$ be a set of points and L a set of affine lines in $\mathbf{F}_p^2$. Assume that all lines are given by an equation $y=ax+b$ with $a\neq 0$ and that all $(u,v)\in P$ satisfy $u\neq 0$.

   (a) Find a large Sidon subset $A\subset \mathbf{F}_p^\times\times\mathbf{F}_p$ and subsets B, $C\subset \mathbf{F}_p^\times\times\mathbf{F}_p$ such that

   $$|\{(b,c)\in B\times C \mid b+c\in A\}| = |\{(p,\ell)\in P\times L \mid p\in\ell\}|.$$

   (Hint: write the equations of the lines in the form $y=ax+b$ and the coordinates of the points as (u,v), and interpret the equation $au+b=v$.)

   (b) Deduce from this and from the previous exercise that

   $$|\{(p,\ell)\in P\times L \mid p\in\ell\}| = \frac{|P||L|}{p} + O(p^{1/2}\sqrt{|P||L|}).$$

   (c) When is this result interesting?

3. Let p be a prime number. Let A_1, A_2 be subsets of $\mathbf{F}_p^\times$ and $A_3\subset\mathbf{F}_p$. Let $G=\mathbf{F}_p^\times\times\mathbf{F}_p$ and consider the subsets

   $$B=\{(x,x) \mid x\in A_1\}\subset G, \qquad C=A_2\times A_3\subset G.$$

   (a) Show that $|B\star C| = |A_1A_2||A_1+A_3|$, where $\star$ refers to the group law in G.

   (b) Find a large Sidon set $A\subset G$ such that $|A\cap B|=|B|$.

   (c) Deduce that there exists a constant $c>0$ such that

   $$\max(|A_1A_2|,|A_1+A_3|) \geqslant c\min((|A_1|p)^{1/2}, |A_1|(|A_2||A_3|p^{-1})^{1/2}).$$

   (d) When does this result imply a non-trivial bound for the classical sum-product problem in $\mathbf{F}_p$?

Note: the results in these exercises are due to Cilleruelo, the last one recovering a previous result of Garaev.