Exercise sheet 6

1. Let G be an abelian group and $A \subset G$ finite. For $k \geqslant 1$, define $\mathcal{F}_k(A)$ to be the size of the largest subset of A which does *not* contain a proper k-term arithmetic progression.

   (a) For abelian groups G and H such that $|G|$ and $|H|$ are prime numbers, and subsets $A \subset G$ and $B \subset H$, show that $\mathcal{F}_k(A)\,\mathcal{F}_k(B) \leqslant \mathcal{F}_k(A \times B)$, with $A \times B \subset G \times H$.

   (b) For $n \geqslant 1$, show that a proper 3-term progression in $\mathbf{F}_3^n$ is an affine line in this $\mathbf{F}_3$-vector space. Moreover, show that such a line ℓ is of the form $\ell = \{x_1, x_2, x_3\}$ where $x_i = (x_{i,1}, \ldots, x_{i,n})$ and for $j = 1, \ldots, n$, *either*

   $$x_{1,j} = x_{2,j} = x_{3,j}$$

   or

   $$\{x_{1,j}, x_{2,j}, x_{3,j}\} = \mathbf{F}_3.$$

   (c) For $n \geqslant 1$, show that $\mathcal{F}_3(\mathbf{F}_3^n) \geqslant 2^n$.

2. Construct an example of a coloring of the set of positive integers in two colors, in such a way that there is no *infinite* arithmetic progression of either color.

3. For positive integers n_0, n and k, we write $P_{n_0,n}(k)$ for the k-term arithmetic progression $\{n_0, n_0 + n, \ldots, n_0 + (k-1)n\}$ in positive integers.

   (a) Let $\gamma > 0$ be a real number. Show that there exists an integer $N_1 \geqslant 1$ with the following property: if $N \geqslant N_1$ and $A \subset [N]$ satisfies $|A| \geqslant \gamma N$, then A contains elements a, b and c with $a + c = 2b$ and $a \neq c$.

   (b) Let A be a set of positive integers. Let $k \geqslant 1$ be an integer and $\gamma > 0$ a real number. Show that there exists an integer $K \geqslant 1$ such that any proper k-term arithmetic progression P of positive integers with $k \geqslant K$ and $|P \cap A| \geqslant \gamma k$ contains a proper 3-term progression which is also contained in A.

   In the remainder of the exercise, we fix a real number $\delta > 0$, an integer $N \geqslant 1$ and a subset $A \subset [N]$ such that $|A| \geqslant \delta N$.

(c) Let $k \geqslant 1$ be an integer. Show that if n is such that $kn < \delta N/k$, then we have

$$\sum_{\substack{n_0 \geqslant 1 \\ n_0+(k-1)n \leqslant N}} |P_{n_0,n}(k) \cap A| \geqslant \delta k\Big(1 - \frac{2}{k}\Big)N.$$

(Hint: for given $a \in A$, show that if $kn \leqslant a \leqslant N - kn$, then a belongs to k among those arithmetic progressions, then estimate how many a satisfy this property.)

(d) For given $n \geqslant 1$, let $\mathcal{G}_n$ be the set of integers $n_0 \geqslant 1$ such that

$$|P_{n_0,n}(k) \cap A| \geqslant \frac{\delta k}{2}.$$

Show that

$$\sum_{\substack{n_0 \geqslant 1 \\ n_0+(k-1)n \leqslant N}} |P_{n_0,n}(k) \cap A| \leqslant \frac{\delta k N}{2} + k|\mathcal{G}_n|.$$

(e) Deduce that if $kn < \delta N/k$ and $k > 8$, then we have

$$|\mathcal{G}_n| \geqslant \frac{\delta N}{4}.$$

(f) Show that the number of values of (n_0, n) such that $|P_{n_0,n}(k) \cap A| \geqslant \delta k/2$ is at least $\delta^2 N^2/(4k^2)$.

(g) Let (a, b, c) be elements of A such that $a + c = 2b$ and $a < c$. Show that if (n_0, n) are such that $\{a, b, c\} \subset P_{n_0,n}(k)$, then n divides $b - a$.

(h) Deduce that the number of (n_0, n) such that $\{a, b, c\} \subset P_{n_0,n}(k)$ is bounded by a constant depending only on k.

(i) Conclude that there exists $N_2 \geqslant 1$ and $c > 0$, depending only on δ, such that if $N \geqslant N_2$ and $|A| \geqslant \delta N$, then A contains at least cN^2 different arithmetic progressions of length 3. (Hint: apply the preceeding results for a value $k = K$ given by an application of (b).)

The result of this exercise is known of *Varnavides's Theorem*; a similar argument applies to Szemerédi's Theorem, and shows that a "weak" statement of existence of at least one k-term progression in any suitably dense set in fact implies the existence of *many* progressions.