Exercise sheet 6

1. Let G be an abelian group and $A \subset G$ finite. For $k \geqslant 1$, define $\mathcal{F}_k(A)$ to be the size of the largest subset of A which does *not* contain a proper k-term arithmetic progression.

   (a) For abelian groups G and H such that $|G|$ and $|H|$ are prime numbers, and subsets $A \subset G$ and $B \subset H$, show that $\mathcal{F}_k(A)\mathcal{F}_k(B) \leqslant \mathcal{F}_k(A \times B)$, with $A \times B \subset G \times H$.

   (b) For $n \geqslant 1$, show that a proper 3-term progression in $\mathbf{F}_3^n$ is an affine line in this $\mathbf{F}_3$-vector space. Moreover, show that such a line ℓ is of the form $\ell = \{x_1, x_2, x_3\}$ where $x_i = (x_{i,1}, \ldots, x_{i,n})$ and for $j = 1, \ldots, n$, *either*

   $$x_{1,j} = x_{2,j} = x_{3,j}$$

   or

   $$\{x_{1,j}, x_{2,j}, x_{3,j}\} = \mathbf{F}_3.$$

   (c) For $n \geqslant 1$, show that $\mathcal{F}_3(\mathbf{F}_3^n) \geqslant 2^n$.

Solution.

(a) First we assume that $k < \min(|G|, |H|)$. Let $\tilde{A} \subset A$ and $\tilde{B} \subset B$ be such that $|\tilde{A}| = \mathcal{F}_k(A)$ and $|\tilde{B}| = \mathcal{F}_k(B)$. We observe that $\tilde{A} \times \tilde{B} \subset A \times B$ does not contain a proper $k-$AP.

Indeed, if it did, then we would have

$$\{(a_0, b_0), (a_0 + a, b_0 + b, \cdots, (a_0 + (k-1)a, b_0 + (k-1)b)\} \subset \tilde{A} \times \tilde{B}.$$

Therefore,

$$\{a_0, a_0 + a, \cdots a_0 + (k-1)a\} \subset \tilde{A}$$
$$\{b_0, b_0 + b, \cdots, b_0 + (k-1)b\} \subset \tilde{B}.$$

We conclude that $\{a_0, a_0 + a, \cdots a_0 + (k-1)a\}$ is a AP with less than k different elements. So, $a_0 + i \cdot a = a_0 + j \cdot a$ for $i \neq 0$, so $(i-j)a = 0$. If $a = 0$ then $\{b_0, b_0 + b, \cdots, b_0 + (k-1)b\}$ has to be a proper k-AP which is a contradiction. Otherwise, since every non-zero element of G has prime order and $k < \min(|G|, |H|)$ we get a contradiction.

A similar argument holds for $k \geqslant \min(|G|, |H|)$.

(b) Recall that an affine line is of the form

$$\{x + a\xi | a \in \mathbf{F}_3\},$$

where $x \in \mathbf{F}_3^n$ and $\xi \in \mathbf{F}_3^n \setminus \{0\}$.

Let $(a_1, \cdots, a_n), (a_1+\tilde{a}_1, \cdots a_n+\tilde{a}_n), (a_1+2\tilde{a}_1, \cdots a_n+2\tilde{a}_n)$ be a proper 3-AP IN $\mathbf{F}_3^n$. Let $x = (a_1, \cdot, a_n), \xi = (\tilde{a}_1, \cdots, \tilde{a}_n)$ and note that $\xi \neq 0$ because the arithmetic progression we are considering in proper. Therefore

$$\{(a_1, \cdots, a_n), (a_1 + \tilde{a}_1, \cdots a_n + \tilde{a}_n), (a_1 + 2\tilde{a}_1, \cdots a_n + 2\tilde{a}_n)\} = \{x + a\xi | a \in \mathbf{F}_3\}.$$

Moreover, observe that if $\tilde{a}_i \in \{1, 2\}$ then $a_i, a_i + \tilde{a}_i, a_i + 2\tilde{a}_i$ are all different elements of $\mathbf{F}_3$. If $\tilde{a}_i = 0$ then $a_i, a_i = \tilde{a}_i = a_i + 2\tilde{a}_i$.

(c) From the previous item, any set of 2 elements is not a $3-$AP in $\mathbf{F}_3$. Writing $\mathbf{F}_3^n = \mathbf{F}_3 \times \cdots \times \mathbf{F}_3$ and using the first item, we conclude that $\mathcal{F}_2(F_3^n) \geqslant 2^n$.

2. Construct an example of a coloring of the set of positive integers in two colors, in such a way that there is no *infinite* arithmetic progression of either colour.

Solution. We colour the positive integers in blocks of incresing size:1 is colored black, 2 and 3 are colored white, $4, 5, 6$ are colored black, and so on.

Suppose $A = \{a_0 + dk, k \in \mathbb{Z}_{>0}\}$ is a infinite AP of one colour and let k be the smallest positive integer such that the block of size kd lies after a_0. Therefore, there exists elements of A in this block. Let a be the largest of such elements. Since the size of the next block is $kd + 1$, it holds that $a_n + d$ is in the next block, there fore a_n and $a_n + d$ have different colours, which is a contradiction.

3. For positive integers n_0, n and k, we write $P_{n_0,n}(k)$ for the k-term arithmetic progression $\{n_0, n_0 + n, \ldots, n_0 + (k-1)n\}$ in positive integers.

(a) Let $\gamma > 0$ be a real number. Show that there exists an integer $N_1 \geqslant 1$ with the following property: if $N \geqslant N_1$ and $A \subset [N]$ satisfies $|A| \geqslant \gamma N$, then A contains elements a, b and c with $a + c = 2b$ and $a \neq c$.

(b) Let A be a set of positive integers. Let $k \geqslant 1$ be an integer and $\gamma > 0$ a real number. Show that there exists an integer $K \geqslant 1$ such that any proper k-term arithmetic progression P of positive integers with $k \geqslant K$ and $|P \cap A| \geqslant \gamma k$ contains a proper 3-term progression which is also contained in A.

In the remainder of the exercise, we fix a real number $\delta > 0$, an integer $N \geqslant 1$ and a subset $A \subset [N]$ such that $|A| \geqslant \delta N$.

(c) Let $k \geqslant 1$ be an integer. Show that if n is such that $kn < \delta N/k$, then we have

$$\sum_{\substack{n_0 \geqslant 1 \\ n_0 + (k-1)n \leqslant N}} |P_{n_0,n}(k) \cap A| \geqslant \delta k \Big(1 - \frac{2}{k}\Big) N.$$

(Hint: for given $a \in A$, show that if $kn \leqslant a \leqslant N - kn$, then a belongs to k among those arithmetic progressions, then estimate how many a satisfy this property.)

(d) For given $n \geqslant 1$, let $\mathcal{G}_n$ be the set of integers $n_0 \geqslant 1$ such that

$$|P_{n_0,n}(k) \cap A| \geqslant \frac{\delta k}{2}.$$

Show that

$$\sum_{\substack{n_0 \geqslant 1 \\ n_0+(k-1)n \leqslant N}} |P_{n_0,n}(k) \cap A| \leqslant \frac{\delta k N}{2} + k|\mathcal{G}_n|.$$

(e) Deduce that if $kn < \delta N/k$ and $k > 8$, then we have

$$|\mathcal{G}_n| \geqslant \frac{\delta N}{4}.$$

(f) Show that the number of values of (n_0, n) such that $|P_{n_0,n}(k) \cap A| \geqslant \delta k/2$ is at least $\delta^2 N^2/(4k^2)$.

(g) Let (a, b, c) be elements of A such that $a + c = 2b$ and $a < c$. Show that if (n_0, n) are such that $\{a, b, c\} \subset P_{n_0,n}(k)$, then n divides $b - a$.

(h) Deduce that the number of (n_0, n) such that $\{a, b, c\} \subset P_{n_0,n}(k)$ is bounded by a constant depending only on k.

(i) Conclude that there exists $N_2 \geqslant 1$ and $c > 0$, depending only on δ, such that if $N \geqslant N_2$, then A contains at least cN^2 different arithmetic progressions of length 3. (Hint: apply the preceeding results for a value $k = K$ given by an application of (b).)

The result of this exercise is known of *Varnavides's Theorem*; a similar argument applies to Szemerédi's Theorem, and shows that a "weak" statement of existence of at least one k-term progression in any suitably dense set in fact implies the existence of *many* progressions.

Solution.

(a) We recall Roth's Theorem:

$$\lim_{n\to\infty} \frac{\mathcal{F}_n(N)}{N} = 0.$$

Let N_1 be such that, for all $N \geqslant N_1$ it holds that

$$\left|\frac{\mathcal{F}_3(N)}{N}\right| < \lambda.$$

therefore, if $|A| \geqslant \lambda N$, then A must contain a proper $3-$AP.

(b) We take $K = N_1$ as in the previous item, and apply the result to $P \cap A$.

(c) Let $a \in A$ and suppose that $kn \leqslant a \leqslant N - kn$. We write

$$\begin{aligned}
a &= a + 0 \cdot n \\
a &= a - n + 1 \cdot n \\
&(\cdots) \\
a &= (a - (k-1)n) + (k-1)n.
\end{aligned}$$

Observe that $a - (k-1)n \geqslant 1$ from the hypothesis. Thefore, there a belogs to k among arithmetic progressions of the form $P_{n_0,n}(k)$. We also observe that

$$|A \cap \{kn \leqslant a \leqslant N - kn\}| \geqslant \delta N - 2kn \leqslant \delta N - \frac{2\delta N}{k},$$

therefore

$$\sum_{\substack{n_0 \geqslant 1 \\ n_0 + (k-1)n \leqslant N}} |P_{n_0,n}(k) \cap A| \geqslant k\left(\delta N - \frac{2\delta N}{k}\right) = \delta k\Big(1 - \frac{2}{k}\Big)N.$$

(d) There are at most N elements n_0 such that $|P_{n_0,n}(k) \cap A| \leqslant \frac{\delta k}{2}$. If $n_0 \in \mathcal{G}_n$ we bound $P_{n_0,n}(k) \cap A$ by k. Therefore,

$$\sum_{\substack{n_0 \geqslant 1 \\ n_0 + (k-1)n \leqslant N}} |P_{n_0,n}(k) \cap A| \leqslant \frac{\delta k N}{2} + k|\mathcal{G}_n|.$$

(e) From the two previous items we have

$$\delta k \left(1 - \frac{2}{k}\right) N \leqslant \frac{\delta k N}{2} + k|\mathcal{G}_n|,$$

so

$$|\mathcal{G}_n| \geqslant \delta N \left(1 - \frac{2}{k} - \frac{1}{2}\right) \geqslant \frac{\delta N}{4},$$

where we used that $k > 8$ in the last inequality.

(f) From the previous item, the number of (n_0, n) such that $|P_{n_0,n}(k) \cap A| \geqslant \delta k/2$ is at least $\delta^2 N^2/(4k^2)$ is

$$\sum_{n \leqslant \frac{\delta N}{k^2}} |\mathcal{G}_n| \geqslant \frac{\delta N}{4} \cdot \frac{\delta N}{k^2} = \frac{\delta^2 N^2}{4k^2}.$$

(g) If $\{a, b, c\} \subset P_{n_0,n}(k)$ then it holds that

$$\begin{aligned} a &= n_0 + l_1 n \\ b &= n_0 + l_2 n \\ c &= n_0 + l_3 n \end{aligned}$$

and $b - a = (l_2 - l_1)n$, so n divides $b - a$.

(h) Fix $\{a, b, c\} \subset [N]$ such that $a < c$ and $a + c = 2n$. From the previous item, the common different of this AP is dn.

If $b - a = dn$ then we must have

$$a = n_0 + jdn$$

for $0 \leqslant j \leqslant k-3$, which implies that there are at most k of such progressions. Also, it holds that $2d < k$ so the number of (n_0, n) such that $\{a, b, c\} \subset P_{n_0,n}(k)$ is bounded by $\frac{1}{2}k(k-2)$.

(i) Take $N_2 = K$ such that for all $k \geqslant K$ such that if P is a proper $k-$AP with $|A \cap P| \geqslant \frac{\delta}{2}k$, then A contains a proper $3-$AP. The existance of such K was proven in item b).

From item f) there exist at least $\frac{\delta^2 N^2}{4k^2}$ values of (n_0, n) such that

$$|P_{n_0,n}(k) \cap A| \geqslant \frac{\delta k}{2}.$$

From h), we are counting each proper $3-$AP at most $\frac{1}{2}k(k-2)$ times, so there exists at least cN^2 proper $3-$AP in A, where c is a constant that depends only on k and δ.