First question
Commutative Algebra: Fall 2023
Exam

The first problem on your exam will be one of the following 10 problems, chosen randomly.

Problem 1. Let R be a ring, $\mathfrak{p}$ a prime ideal, and $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ maximal ideals such that $\prod_{i=1}^n \mathfrak{m}_i = 0$. Show that $\mathfrak{p} = \mathfrak{m}_i$ for some i.

Problem 2. Let R be an integral domain, I a nonzero ideal in R, and $K = \operatorname{Frac}(R)$. Show that $I \otimes_R K \cong K$.

Problem 3. Let R be a ring, $S \subset R$ a multiplicative subset, and M a finitely generated R-module. Suppose that $S^{-1}M = 0$. Show that there is an $f \in S$ such that $M_f = 0$, where M_f is the localization of M at $\{f^n : n \in \mathbb{N}\}$.

Problem 4. Let m and n be natural numbers. Show that there is a short exact sequence

$$0 \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

For which m, n does the exact sequence split?

Problem 5. Let m and n be natural numbers. Compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.

Problem 6. Let R be a ring, $S \subset R$ a multiplicative subset. Let $I = \{a \in R : as = 0 \text{ for some } s \in S\}$. Show that I is an ideal, and then show that the localization map $R \to S^{-1}R$ factors through R/I.

Problem 7. Let $R = \mathbb{C}[x, y]/(y^2 - x^3)$. Let $K = \operatorname{Frac}(R)$. Compute the integral closure of R in K.

Problem 8. Let R be an Artinian ring. Prove that if R is a domain, then it is a field.

Problem 9. Let k be a field. Compute the dimension of $k[x]_x$, the localization of $k[x]$ at the multiplicative subset $S = \{x^n : n \in \mathbb{N}\}$.

Problem 10. Let k be a field, $d \in \mathbb{N}$. Compute the Hilbert function and dimension of

$$k[x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d],$$

the subalgebra of $k[x, y]$ generated by all degree d monomials.