Due Date: 29 Sept
Commutative Algebra: Fall 2023
Exercise sheet 1

Problem 1. (Gathmann exercise 1.13) Show that the equation of ideals

$$(x^3 - x^2, x^2y - x^2, xy - y, y^2 - y) = (x^2, y) \cap (x - 1, y - 1)$$

holds in the polynomial ring $\mathbb{C}[x, y]$. Is this a radical ideal? What is the vanishing locus of the ideal in $\mathbb{A}^2_{\mathbb{C}}$?

Problem 2. Let R be a ring. An element $e \in R$ such that $e^2 = e$ is called an *idempotent*. An idempotent $e \in R$ is called trivial if $e = 0$ or $e = 1$.

1. Prove that R is a product of two nontrivial rings if and only if R has a nontrivial idempotent.
2. We say a ring R is *local* if it has a unique maximal ideal. Show that if $e \in R$ is an idempotent in a local ring R, then e is trivial.

Problem 3. (Gathmann exercise 2.23(a)) Let R be a ring and I an ideal of R. We say a prime ideal $\mathfrak{p}$ containing I is *minimal* over I if for every prime ideal $\mathfrak{q}$ such that

$$I \subset \mathfrak{q} \subset \mathfrak{p},$$

we have $\mathfrak{q} = \mathfrak{p}$. Prove that for every ideal I in R there exists a minimal prime $\mathfrak{p}$.

Problem 4. Let k be a field, $A = k[x, y]/(xy - 1)$, and $B = k[z]$. Show that any morphism of k-algebras $\varphi : A \to B$ maps x to a constant $c \in k$.