Due Date: 24 Nov
Commutative Algebra: Fall 2023
Exercise sheet 10

Problem 1. Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a $\mathbb{N}$-graded ring. Prove that any homogeneous maximal ideal $\mathfrak{m}$ of R is of the form $\mathfrak{n} \oplus \bigoplus_{i \geq 1} R_i$ for $\mathfrak{n}$ a maximal ideal of R_0.

Problem 2. Let $R \to S$ be a ring map and M an S module. Via the map $R \to S$, M is also an R module. Prove $\ell_S(M) \leq \ell_R(M)$. Prove that equality holds if $R \to S$ is surjective.

Problem 3. Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a $\mathbb{N}$-graded ring and M a finitely generated graded module over R. Prove that $\ell_R(M) = \ell_{R_0}(M) = \sum_i \ell_{R_0}(M_i)$.

Problem 4. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated modules over a Noetherian ring R. Prove that $\dim(M)$ is the maximum of $\dim(M')$ and $\dim(M'')$.