Due Date: 6 Oct
Commutative Algebra: Fall 2023
Exercise sheet 2

Problem 1. Let $(R, \mathfrak{m})$ be a local ring and M a finite R module. Let $N \to M$ be a map of R modules such that the induced map $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is surjective. Show that $N \to M$ is surjective.

Problem 2. Let R be a ring. Prove the snake lemma: if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & M & \xrightarrow{\varphi} & N & \xrightarrow{\psi} & P & \longrightarrow & 0 \end{array}$$

is a commutative diagram of R modules with exact rows, then there is an exact sequence

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to 0.$$

(Recall that the cokernel of a morphism $h : M_1 \to M_2$ of R modules is $\operatorname{coker}(h) = M_2/\operatorname{im}(h)$.) If you get stuck, you can watch this scene from the movie *It's My Turn*.

Problem 3. Let R be a ring

(i) Show that if

$$0 \to M_1 \to M_2 \to M_3$$

is an exact sequence of R modules, then so is

$$0 \to \operatorname{Hom}(N, M_1) \to \operatorname{Hom}(N, M_2) \to \operatorname{Hom}(N, M_3).$$

Make sure you first understand what the maps in the sequence are!

(ii) Come up with a counterexample to the following statement: if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of R modules, then so is

$$0 \to \operatorname{Hom}(N, M_1) \to \operatorname{Hom}(N, M_2) \to \operatorname{Hom}(N, M_3) \to 0.$$

(iii) We say that an R module P is *projective* if for every exact sequence of R modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

the induced sequence

$$0 \to \operatorname{Hom}(P, M_1) \to \operatorname{Hom}(P, M_2) \to \operatorname{Hom}(P, M_3) \to 0$$

is exact. Show that the following are equivalent.

- P is projective.
- P is a direct summand of a free module: there exists an R module N such that $P \oplus N$ is free.
- Every exact sequence of R modules

  $$0 \to M \to N \to P \to 0$$

  splits.