Due Date: 10 Nov
Commutative Algebra: Fall 2023
Exercise sheet 7

Problem 1. Let R be a ring. Show that a closed subset $Z \subset \mathrm{Spec}(R)$ is irreducible if and only if $Z = V(\mathfrak{p})$ for a prime $\mathfrak{p}$.

Problem 2. Let R be a ring, M an R module, and $N \subset M$ a submodule. Show that $\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(M/N)$.

Problem 3. A topological space X is called Noetherian if its open subsets satisfy the ascending chain condition (any chain of open subset $U_0 \subset U_1 \subset \dots$ eventually stabilizes). Show that if a ring R is Noetherian, then $\mathrm{Spec}(R)$ is a Noetherian topological space. What about the converse?

Problem 4. Let R be a ring. Show that $\mathrm{Spec}(R)$ is irreducible if and only if the nilradical $\sqrt{(0)}$ is prime.