Problem 1.

The following lemma will be useful for our purposes:

Lemma 0.1. *If R is a commutative ring and $I, J \subset R$ are ideals such that $I + J = R$ then $I \cap J = IJ$.*

Proof. Indeed, it is always the case that $IJ \subset I \cap J$ and from our assumption we have

$$I \cap J = (I + J) \cdot (I \cap J) = I \cdot (I \cap J) + J \cdot (I \cap J) \subset IJ + JI = IJ.$$

Thus $IJ = I \cap J$. □

Now as $y - (y - 1) = 1$ we can apply the lemma to the ideals (x^2, y) and $(x - 1, y - 1)$ in $\mathbb{C}[x, y]$ so

$$(x^2, y) \cap (x-1, y-1) = (x^2, y) \cdot (x-1, y-1) = (x^2(x-1), x^2(y-1), y(x-1), y(y-1)).$$

This ideal is not radical as the element $(x(x-1))^2$ belongs to it whereas $x(x-1)$ does not as it does not belong to the ideal (x^2, y).

Problem 2.1.

Recall the Chinese remainder theorem:

Theorem 0.2. *If R is a commutative ring and $I, J \subset R$ are ideals such that $I + J = R$ then $R/IJ \simeq R/I \oplus R/J$.*

If R has a nontrivial idempotent e we have $e(e - 1) = 0$. Neither e nor $e - 1$ is invertible, otherwise multiplying the identity by the corresponding inverse would yield $e = 0$ ore $e - 1 = 0$ contradicting the assumption. So we have

$$R \simeq R/(0) = R/(e(e - 1)) = R/(e) \oplus R/(e - 1)$$

yielding a non-trivial decomposition of R into a direct product of two rings. If $R = R_1 \oplus R_2$ an element $(0, 1)$ is a non-trivial idempotent.

Problem 2.2.

Denote by $\mathfrak{m}$ the maximal ideal of R. Assume there is a non-trivial idempotent $e \in R$. As we've seen $e - 1$ as well as e is not invertible, thus $e - 1 \in \mathfrak{m}$ and $e \in \mathfrak{m}$, hence $1 = e - (e - 1) \in \mathfrak{m}$ leading to a contradiction.

Problem 3.

Denote by X the set of all primes containing I. It is naturally a poset ordered by inclusion and X is non-empty as every ideal is contained in a maximal ideal which is prime.

Let us verify that X satisfies the condition of Zorn's lemma. Indeed if $\{\mathfrak{p}_i\}$ is a chain in X then $J := \bigcap_i \mathfrak{p}_i$ is an ideal containing I. Let us verify that it is prime: take $a, b \in R$ such that $ab \in J$. Assume that $a, b \notin J$. Then we can find i, j such that $a \notin \mathfrak{p}_i, b \notin \mathfrak{p}_j$. But as $\{\mathfrak{p}_k\}$ form a chain we have $\mathfrak{p}_i \subset \mathfrak{p}_j$ or

$\mathfrak{p}_j \subset \mathfrak{p}_i$ therefore $a, b \notin \mathfrak{p}_i$ or $a, b \notin \mathfrak{p}_j$ hence the same is true for ab as $\mathfrak{p}_k$ are prime ideals. This is a contradiction.

Applying Zorn's lemma to X yields a minimal prime containing I.

Problem 4.

As $xy = 1 \in A$ we have $\phi(x)\phi(y) = 1 \in B$. In particular $\phi(x)$ is invertible, hence it should be a degree 0 polynomial in $k[x]$ i.e. a constant.