Problem 1.

For any homogeneous ideal $I \trianglelefteq R$ we have

$$I = \bigoplus_{i=0}^{\infty} I \cap R_i \subseteq I \cap R_0 \oplus \bigoplus_{i=1}^{\infty} R_i,$$

where the latter is an ideal. Moreover if we pick a maximal ideal $\mathfrak{n}$ of R_0 containing $I \cap R_0$ we have

$$I \cap R_0 \oplus \bigoplus_{i=1}^{\infty} R_i \subseteq \mathfrak{n} \oplus \bigoplus_{i=1}^{\infty} R_i$$

and

$$R/(\mathfrak{n} \oplus \bigoplus_{i=1}^{\infty} R_i) \simeq R_0/\mathfrak{n}$$

Hence, we checked that any homogeneous ideal is contained in an ideal of the form $\mathfrak{n} \oplus \bigoplus_{i=1}^{\infty} R_i$ with a maximal $\mathfrak{n} \trianglelefteq R_0$ and ideals of such form are maximal.

Problem 2.

Any chain of S-submodules of M is automatically a chain of R-submodules so the inequality follows. Moreover if $R \to S$ is surjective then any R-submodule of M is stable under the action of S, so any chain of R-submodules is automatically a chain of S-submodules and the inverse inequality follows.

Problem 3.

Consider the augmentation morphism $\pi \colon R \twoheadrightarrow R/(R_{>0}) \simeq R_0$. Let $M = \bigoplus_{i \geq 0} M_i$ be the decomposition of M into homogeneous parts. One can see that

$$F^i(M) \coloneqq \bigoplus_{j \geq i} M_j$$

is an R-submodule of M for any i, so we have a chain

$$M = F^0 M \supset F^1 M \supset \cdots$$

so it follows that

$$\ell_R(M) = \sum_{i \geq 0} \ell_R(F^i M / F^{i+1} M).$$

As $R^j \cdot M^i \subset M^{i+j}$, it follows that $\operatorname{ann}_R(F^i M/F^{i+1}M) \supset R_{>0}$. So the action of R on all successive quotients F^i/F^{i+1} factors through π. It follows from problem 2 that

$$\ell_R(F^i M/F^{i+1}M) = \ell_{R_0}(F^i M/F^{i+1}M).$$

$F^i M/F^{i+1}M \simeq M_i$ as R_0-modules, so we have $\ell_{R_0}(F^i M/F^{i+1}M) = \ell_{R_0} M_i$ and the equality follows.

Problem 4.

We have an equality

$$\text{Supp}\,(M) = \text{Supp}\,(M') \cup \text{Supp}\,(M'')$$

of two closed subsets of $\text{Spec}(R)$. As $\dim N = \dim(\text{Supp}\,(N))$ it follows that

$$\dim(M) = \max\{\dim(M'), \dim(M'')\}.$$