Problem 1.

By Gathmann, Corollary 12.17 every nonzero ideal of R is of the form P^n, where P is the unique maximal ideal of R. So there are two prime ideals, namely P and $\{0\}$. The former is a closed point in $\mathrm{Spec}(R)$ as P is maximal, and the latter is a non-closed point in $\mathrm{Spec}(R)$. This uniquely describes the topology on the set $\mathrm{Spec}(R)$ of cardinality 2.

Problem 2.

Suppose ν is a discrete valuation on k. Pick an element $x \in k$, such that $\nu(x) = 1$. As k is algebraically closed there exists $y \in k$, such that $y^2 = x$. Then we have $1 = \nu(x) = 2\nu(y)$, contradiction.

Problem 3.

We use the characterisation of valuation rings as in Gathmann, Proposition 12.8. It is clear that $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ are integral domains.

Now we need to check that for any $a \in \mathrm{Frac}(R_{\mathfrak{p}})\backslash\{0\}$ one of a and $\frac{1}{a}$ lies in $R_{\mathfrak{p}}$. Observe that $\mathrm{Frac}(R_{\mathfrak{p}}) = \mathrm{Frac}(R)$, so one of a and $\frac{1}{a}$ lies in R and hence in $R_{\mathfrak{p}}$.

Now we need to check that for any $b \in \mathrm{Frac}(R/\mathfrak{p})\backslash\{0\}$ one of b and $\frac{1}{b}$ lies in $R/\mathfrak{p}$. Write b as $\frac{x}{y}$ with $x, y \in (R/\mathfrak{p})\backslash\{0\}$, lift x, y to $x_1, y_1 \in R$. If $\nu_R(x_1) \geq \nu_R(y_1)$, then $\frac{x_1}{y_1} \in R$ and $b = \frac{x_1}{y_1} \mod \mathfrak{p}$ lies in $R/\mathfrak{p}$. Otherwise $b^{-1} \in R/\mathfrak{p}$.