Problem 1.

Denote the morphism by ϕ. It follows from the assumptions that the module $P := N/\mathrm{Im}(\phi)$ is finitely generated and satisfies $P = \mathfrak{m}P$. It follows from Nakayama's lemma (Corollary 3.27 in the Gathmann's textbook) that there is $a \in \mathfrak{m}$, such that $(1-a)|_P = 0$. But $(R, \mathfrak{m})$ is local, hence $1-a$ is invertible, therefore $P = 0$, i.e. $\mathrm{Im}(\phi) = N$, which is equivalent to ϕ being surjective.

Problem 2.

See the proof of Lemma 4.7 in the Gathmann's textbook.

Problem 3.1.

Denote the morphisms in the exact sequence by $\phi\colon M_1 \to M_2$ and $\psi\colon M_2 \to M_3$. Note that exactness is equivalent to saying that ϕ is the kernel of ψ. The arrows in the Hom-sequence correspond to composition with ϕ and ψ respectively.

Let us show exactness of the Hom-sequence in the middle term: suppose $\theta\colon N \to M_2$ satisfies $\psi \circ \theta = 0$. Then θ should factor through the kernel of ψ which is ϕ, i.e. there is a morphism $\eta\colon N \to M_1$, s.t. $\theta = \phi \circ \eta$.

Now let us show exactness of the Hom-sequence in the first term: suppose $\eta\colon N \to M_1$ satisfies $\phi \circ \eta = 0$. Then by injectivity of ϕ we have $\eta = 0$.

Problem 3.2. Take $R = \mathbb{Z}$ and the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Applying $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z})$ to the sequence we obtain the sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0,$$

which is not exact in the right $\mathbb{Z}/2\mathbb{Z}$ term.

Problem 3.3.

$1 \implies 3$: Suppose P is projective. Given a short exact sequence

$$0 \to M \to N \xrightarrow{\phi} P \to 0$$

applying $\mathrm{Hom}(P, -)$ we get a short exact sequence

$$0 \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \xrightarrow{\phi_*} \mathrm{Hom}(P, P) \to 0.$$

In particular the element $\mathrm{id}_P \in \mathrm{Hom}(P, P)$ can be lifted to $\mathrm{Hom}(P, N)$, i.e. there is a morphism $\psi\colon P \to N$, such that $\phi \circ \psi = \mathrm{id}_P$ giving a splitting of the initial short exact sequence.

$3 \implies 2$: Suppose any short exact sequence

$$0 \to M \to N \xrightarrow{\phi} P \to 0$$

splits. Any module has a surjection from a free one so take any such surjection $R^{\oplus I} \xrightarrow{\psi} P$ and denote by $K \hookrightarrow R^{\oplus I}$ its kernel. Then we have a short exact sequence

$$0 \to K \to R^{\oplus I} \xrightarrow{\psi} P \to 0$$

which splits by assumption providing an isomorphism $K \oplus P \simeq R^{\oplus I}$.

$2 \implies 1$: Note that $\mathrm{Hom}(R^{\oplus I}, M) = M^{\oplus I}$ and

$$\mathrm{Hom}(R^{\oplus I}, (M \xrightarrow{\phi} N)) = M^{\oplus I} \xrightarrow{\phi^{\oplus I}} N^{\oplus I},$$

so any free module is projective. Also we have

$$\mathrm{Hom}(N \oplus P, -) = \mathrm{Hom}(N, -) \oplus \mathrm{Hom}(P, -),$$

thus a direct summand of a projective module is again projective (as a direct sum of two sequences is exact iff they are both exact).