Problem 1.

We claim that $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} \simeq 0$.

Indeed, $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ is generated by the tensors $\alpha \otimes \beta$, where $\alpha, \beta \in \mathbb{Q}/\mathbb{Z}$. Write $\frac{p}{q}$ and $\frac{m}{n}$ with integer p, q, m, n for representatives of α, β in $\mathbb{Q}$. We have

$$\frac{p}{q} \otimes \frac{m}{n} = \frac{np}{nq} \otimes \frac{m}{n} = n\frac{p}{nq} \otimes \frac{m}{n} = \frac{p}{nq} \otimes n\frac{m}{n} = \frac{p}{nq} \otimes m.$$

But $m = 0$ in $\mathbb{Q}/\mathbb{Z}$ hence $\frac{p}{nq} \otimes m = 0$, therefore any generator of $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}$ is 0 hence the module is 0.

We claim that the product map

$$\phi\colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}, \alpha \otimes \beta \mapsto \alpha\beta$$

is an isomorphism. Indeed, similarly to the previous computation we have $\frac{p}{q} \otimes \frac{m}{n} = \frac{p}{nq} \otimes m = \frac{pm}{nq} \otimes 1$ i.e. any bilinear map $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \to M$ factors through ϕ.

We have

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\phi} \mathbb{C}[x]/(x^2+1) \simeq \mathbb{C}/((x+i)(x-i)) \simeq \mathbb{C} \oplus \mathbb{C},$$

the last isomorphism following from the Chinese remainder theorem and the map ϕ given by $[f(x)] \otimes \alpha \mapsto [\alpha f(x)]$. The latter is correctly defined and as $x^2+1|f \Leftrightarrow x^2+1|\alpha f$ and has an inverse given in $\mathbb{R}$-basis by

$$[1] \mapsto [1] \otimes 1; [i] \mapsto [1] \otimes i; [x] \mapsto [x] \otimes 1; [ix] \mapsto [x] \otimes i.$$

Therefore ϕ is an isomorphism, hence $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$.

We claim that $\phi\colon \mathbb{Q}[x] \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}[x]$ given by $f(x) \otimes \alpha \mapsto \alpha f(x)$ is an isomorphism. Indeed, the inverse is given by $\alpha x^i \mapsto x^i \otimes \alpha$.

Problem 2.1.

Given a short exact sequence $0 \to P \to Q \to S \to 0$, tensoring with $\bigoplus_{i \in I} M_i$ we obtain the direct sum of sequences $0 \to P \otimes M_i \to Q \otimes M_i \to S \otimes M_i \to 0$, whose exactness is equivalent to exactness of each of the summand, therefore the assertion follows.

Problem 2.2. Applying the previous problem to the free rank 1 modules $M_i = R$ we deduce that any free module is flat. Applying the problem again we deduce that any direct summand of a free module is flat, i.e. any projective module is flat.

Problem 2.3.

Consider $R = \mathbb{Z}$ and take $N = \mathbb{Q}$.

Recall that for any $\mathbb{Z}$-module V we can explicitely describe $V \otimes_{\mathbb{Z}} \mathbb{Q}$ as a localization, namely $V \otimes \mathbb{Q} = V_{\mathbb{Q}} := \{\frac{v}{r} | v \in V, r \in \mathbb{Z}\}/\sim$, where $\sim$ is the equivalence relation

$$\frac{v}{r} \sim \frac{v'}{r'} \Leftrightarrow \exists n \in \mathbb{Z} : n(r'v - rv') = 0.$$

Now let us firstly verify that N is flat. Given an injective map $\phi\colon M \hookrightarrow M'$ of $\mathbb{Z}$-modules, suppose that the induced map $\phi_{\mathbb{Q}}\colon M \otimes \mathbb{Q} \to M' \otimes \mathbb{Q}$ is not injective. Take a non-zero element $\frac{m}{q} \in \mathrm{Ker}(\phi_{\mathbb{Q}})$. As $\frac{m}{q}$ is nonzero, then $m \in M$ is non-torsion, otherwise if $rm = 0$ then $m \otimes \frac{1}{q} = rm \otimes \frac{1}{rq} = 0$. By injectivity of ϕ it follows that $\phi(m) \in M'$ is non-torsion as well. But then $\frac{\phi(m)}{1}$ is a non-zero element of $M'_{\mathbb{Q}}$, thus $m \otimes 1$ and hence $\frac{m}{q}$ cannot belong to the kernel of $\phi_{\mathbb{Q}}$, a contradiction.

We are left to verifying that N is not projective. Indeed, otherwise it would be a direct summand of $\mathbb{Z}^{\oplus I}$ but this module does not contain any non-trivial divisible elements.

Problem 3.

To extend α set $\alpha(\frac{r}{s}) := \alpha(s)^{-1}\alpha(r)$, it follows from the construction of the localization that this is correctly defined. Now to verify uniqueness of the extension use that $r = s \cdot \frac{r}{s} \in S^{-1}R$, therefore it should satisfy

$$\alpha(r) = \alpha(s)\alpha(\frac{r}{s}) \in R'.$$

As $\alpha(s)$ is invertible in R', multiplying by its inverse we have

$$\alpha(\frac{r}{s}) = \alpha(s)^{-1}\alpha(r).$$

Problem 4a.

Pick some elements $m_i \in M_i$. We have

$$\frac{m_1 + m_2}{s} = \frac{m_1}{s} + \frac{m_2}{s},$$

hence $S^{-1}(M_1 + M_2) \subset S^{-1}M_1 + S^{-1}M_2$. Also we have

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2},$$

hence $S^{-1}M_1 + S^{-1}M_2 \subset S^{-1}(M_1 + M_2)$.

Problem 4b.

If $m \in M_1 \cap M_2$ then $\frac{m}{s}$ lies in both $S^{-1}M_1$ and $S^{-1}M_2$, hence

$$S^{-1}(M_1 \cap M_2) \subset S^{-1}M_1 \cap S^{-1}M_2$$

and this inclusion holds for arbitrary intersections. Now consider

$$v = \frac{m_1}{s_1} = \frac{m_2}{s_2} \in S^{-1}M_1 \cap S^{-1}M_2.$$

By the definition of localization there is $s \in S$ such that

$$s(s_2 m_1 - s_1 m_2) = s s_2 m_1 - s s_1 m_2 = 0 \in M,$$

in particular $s s_2 m_1 \in M_1 \cap M_2$ and hence $v = \frac{s s_2 m_1}{s s_2 s_1} \in S^{-1}(M_1 \cap M_2)$ which proves the converse inclusion.

For a counterexample consider $M_i = (i) \subset \mathbb{Z}$ as $\mathbb{Z}$-modules and $S = \mathbb{Z} \setminus \{0\}$. We have $\cap_i M_i = \{0\}$, however $S^{-1} M_i = \mathbb{Q} = S^{-1}\mathbb{Z}$ and $\cap_i S^{-1} M_i = \mathbb{Q}$.

Problem 4c.

Pick an element $a \in \sqrt{I}$. We have $a^n \in I$ for some n, hence $(\frac{a}{s})^n = \frac{a^n}{s^n} \in S^{-1} I$, therefore $S^{-1}\sqrt{I} \subset \sqrt{S^{-1} I}$.

Pick an element $\frac{b}{s} \in \sqrt{S^{-1} I}$. We have $(\frac{b}{s})^n \in S^{-1} I$ for some n, i.e. $\frac{b^n}{s^n} = \frac{a}{r} \in S^{-1} R$ for some $a \in I, r \in S$. Therefore

$$t(r b^n - s^n a) = 0 \in R$$

for some $t \in S$, in particular $t r b^n \in I$ and $\frac{b}{s} = \frac{trb}{trs} \in S^{-1}\sqrt{I}$ as $(trb)^n = (tr)^{n-1} t r b^n \in I$. This proves the converse inclusion.

Problem 5a.

Suppose $I \subset J$, i.e. there is an injective map $I \hookrightarrow J \subset R$ of R-modules. As localization is exact, the map $I_P \to J_P$ is injective as well.

Now suppose for any maximal ideal $P \subset R$ the ideal I_P is contained in J_P. We need to verify that $I \subset J$ i.e. that the composite $I \hookrightarrow R \to R/J$ vanishes. Consider the following diagram:

$$\begin{array}{ccc} I & \longrightarrow & R/J \\ \downarrow & & \downarrow \\ \prod_P I_P & \xrightarrow{0} & \prod_P (R/J)_P \end{array} \tag{0.1}$$

where the lower map is 0 being the composite of

$$\prod I_P \to \prod R_P \to R_P / J_P = \prod (R/J)_P.$$

The last equality is the natural identification following from exactness of localization.

Now we use that for any module M the natural map $M \to \prod_P M_P$ is injective. In particular so is the right vertical arrow of the diagram 0.1 implying that the upper horizontal arrow is 0.

Problem 5b.

Applying Problem 4c to the zero ideal we obtain that $Nil(R)_P = Nil(R_P)$.

Now apply the previous problem to the nilradical of R.

Problem 5c.

Consider the ring $\mathbb{Q} \times \mathbb{Q}$. It is not an integral domain as $(0,1) \cdot (1,0) = 0$, but the localizations at maximal ideals $(\mathbb{Q}, 0)$ and $(0, \mathbb{Q})$ are integral domains being $\mathbb{Q}$. Geometrically localization detects local properties at the point and does not know about connected components which do not contain this point.