Problem 1.

We need to show that $M/IM = 0$. Viewed as R-module it is annihilated by I and thus naturally aquires a structure of a R/I-module. To show that it is a zero module let us show that $(M/IM)_{\mathfrak{m}} = 0$ for all maximal ideals of $\mathfrak{m} \trianglelefteq R/I$. The maximal ideals of R/I are in one-to-one correspondence with maximal ideals of R containing I. Pick a maximal ideal $\mathfrak{m} \trianglelefteq R/I$ and let $\tilde{\mathfrak{m}} \trianglelefteq R$ be the preimage of $\mathfrak{m}$ w.r.t. reduction map $R \to R/I$. Note that for any R/I-module N viewed as an R-module via $R \to R/I$ we have

$$N_{\tilde{\mathfrak{m}}} \simeq N_{\mathfrak{m}}.$$

Hence

$$(M/IM)_{\mathfrak{m}} \simeq (M/IM)_{\tilde{\mathfrak{m}}} \simeq M_{\tilde{\mathfrak{m}}}/(IM)_{\tilde{\mathfrak{m}}} = 0$$

as R-modules, the last isomorphism following from exactness of localizations.

□

Problem 2a.

The element is not integral over $\mathbb{Z}$.

Consider the chain

$$\mathbb{Z} \subset \mathbb{Z}[\sqrt{2}] \subset \mathbb{Z}[\sqrt{2}][\sqrt{2+\sqrt{2}}]$$

of rings. Every extension is finite being an extension of the form $R \to R[x]/(p(x))$ with a monic polynomial $p(x)$. Therefore the composite is finite and hence integral. So the element $\sqrt{2+\sqrt{2}}$ is integral over $\mathbb{Z}$. Now if $\sqrt{2+\sqrt{2}} + \frac{1}{2}\sqrt[3]{3}$ were integral over $\mathbb{Z}$ so would be its difference with $\sqrt{2+\sqrt{2}}$ which is equal to $\frac{1}{2}\sqrt[3]{3}$. And if the latter element were be integral, so would be its cube which is equal to $\frac{3}{8}$. But this number is not integral over $\mathbb{Z}$.

Problem 2b.

The extension $R \subset R'$ is integral as R' is generated over R by the element x which satisfies $f(x) = 0$ with monic $f[t] = t^2 - 1 - (x^2 - 1) \in R[t]$.

Now consider $P' = (x - 1) \trianglelefteq R'$. The intersection $P' \cap R$ consists of all polynomials in $x^2 - 1$ which are divisible by $x - 1$ as polynomials in x. This is the same as all polynomials of the form $\sum_{i=1}^{k} a_i(x^2-1)^i$. Assume $R_P \subset R'_{P'}$ is integral, consider the element $y = \frac{1}{x+1} \in R'_{P'}$. If this element were integral over R_P we would have an identity

$$y^n + p_n y^{n-1} + \cdots + p_0 = 0$$

in $R'_{P'}$, where $p_n \in R_P$. In particular the denominators of $\{p_i\}$ are of the form $a_0 + \sum_{i=1}^{k} a_i(x^2-1)^i$ where $a_0 \neq 0$ by considerations above. Myltiplying by

$(x+1)^n$ we obtain

$$1 + p_n(x+1) + \cdots + p_0(x+1)^n = 0.$$

This identity may be considered in the ring $R'_{R\cap P'}$, where the ideal $(x+1)$ does not become the unit ideal as the multiplicative system $R \cap P'$ does not intersect with $(x+1)$ in R'. Reducing mod $(x+1)$ we obtain a contradiction.

Problem 3.

Consider a morphism $\phi\colon R \to \mathbb{R}[z]$ defined by $x \mapsto z^2+1, y \mapsto z(z^2+1)$. It naturally induces a morphism,

$$\tilde{\phi}\colon R[x^{-1}] \to \mathbb{R}[z][(z^2+1)^{-1}]$$

and

$$z = \frac{\tilde{\phi}(y)}{\tilde{\phi}(x)},$$

hence $\tilde{\phi}$ is surjective. But $\tilde{\phi}$ has an inverse given by $z \to \frac{y}{x}$, hence $\tilde{\phi}$ is an isomorphism. In particular ϕ induces an isomorphism of quotient fields so let us identify them via this morphism.

Under the identification z maps to $\frac{y}{x} = t$ so $\mathbb{R}[z]$ maps isomorphically to $R[t]$ as the latter algebra is generated by t over R. In particular $R[t] \simeq \mathbb{R}[t]$ and is a normal ring. Hence $R[t]$ is the normalization of R in K as t is integral over R.

Geometrically the embedding $R \hookrightarrow R[t]$ corresponds to the map

$$\phi^*\colon \mathbb{A}^1 \to V(y^2 - x^2 - x^3)$$

given in coordinates as $z \mapsto (z^2+1, z(z^2+1))$.