Problem 1.

By Chinese remainder theorem we have $\mathbb{C}[x,y]/(y(y-1)) \simeq \mathbb{C}[x] \oplus \mathbb{C}[x]$ via $[f(x,y)] \mapsto (f(x,0), f(x,1))$. Thus

$$R' = \mathbb{C}[x,y]/(y(y-1), xy) \simeq (\mathbb{C}[x] \oplus \mathbb{C}[x])/((0,x)) \simeq \mathbb{C}[x] \oplus \mathbb{C}$$

via the map $[f(x,y)] \mapsto (f(x,0), f(0,1))$, so the morphism $\mathbb{C}[x] = R \to R'$ is given by $g(x) \mapsto (g(x), g(0))$ and R' is generated over R by $(1,0)$ and $(0,1)$ which both satisfy the monic equation $t^2 - t = 0$.

So $R \to R'$ is an integral extension. Now consider the ideal $Q' \trianglelefteq R'$ generated by $(1,0) \in R'$, which is the element corresponding to $1-y$ in the presentation $\mathbb{C}[x,y]/(y(y-1), xy)$. In other words $Q' = \{(f(x), 0)\} \trianglelefteq R'$. We have

$$Q \coloneqq Q' \cap R = \{g(x) | g(0) = 0\} = (x) \trianglelefteq R.$$

Now consider the ideal $P \coloneqq (0) \trianglelefteq R$ which is prime and contained in Q. We claim that in the following diagram

$$\begin{array}{lccccc} & R': & ?? & \subset & Q' \\ (0.1) & & \big\downarrow & & \big\downarrow \\ & R: & P = (0) & \subset & Q = (x) \end{array}$$

There is no prime ideal of R' contained in Q' which lies over P.

Indeed, let $I = \{(h(x), 0)\} \subset Q'$ be an ideal lying over (0). We have

$$I \cap R = \{h(x) | h(0) = 0\}.$$

So if $0 \neq I \ni (h(x), 0)$ then $xh(x) \in I \cap R$ thus the only ideal contained in Q' lying over (0) is the ideal (0) which is not prime in R'.

So the extension $R \hookrightarrow R'$ does not satisfy going down property, but we should not panic as R' is not an integral domain.

Problem 2.

R' is an integral domain therefore R is. Moreover R' is generated over R by the element x which is integral over R as it satisfies $f(x) = 0$ for $f = t^2 - t + x(x-1)$.

Let us find $\mathfrak{q} \coloneqq (1-x, y) \cap R$: any element of R is a linear combination of $(x(1-x))^i y^j (xy)^k$. The element $(x(1-x))^i y^j (xy)^k$ lies in $(1-x, y)$ unless $i = j = k = 0$ so $(1-x, y) \cap R$ is the set of linear combinations of $(x(1-x))^i y^j (xy)^k$ with one of i, j, k greater than 0. I.e. $\mathfrak{q} = (x(1-x), y, xy)$.

Let us find $\mathfrak{p} \coloneqq (x) \cap R$: a linear combination of $\{(x(1-x))^i y^j (xy)^k\}$ lies in (x) unless there is a nonzero term with $i = k = 0$ so $(x) \cap R$ is the set of linear combinations of $(x(1-x))^i y^j (xy)^k$ with one of i, k greater than 0. I.e. $\mathfrak{p} = (x(1-x), xy)$.

It follows from the descriptions above that $\mathfrak{p} \subsetneq \mathfrak{q}$.

Let us show that there is no prime ideal $P \subset (1-x, y)$ contracting to $\mathfrak{p}$. Indeed, otherwise such P should contain $x(x-1)$ and xy. Consider

$$\phi \colon R \to R/(x(x-1), xy) \simeq \mathbb{C}[y] \oplus \mathbb{C}; f(x,y) \mapsto (f(0,y), f(1,0))$$

which is essentially the map from Problem 1 with x and y intertwined.

Primes of R containing $(x(x-1), xy)$ correspond bijectively via ϕ^{-1} to primes of $\mathbb{C}[y] \oplus \mathbb{C}$. Primes of $\mathbb{C}[y] \oplus \mathbb{C}$ are of the form $((1,0))$ or $Q \oplus \mathbb{C}$ where $Q \trianglelefteq \mathbb{C}[y]$ is a prime.

Observe that $\phi^{-1}((1,0)) = \{f \in R | f(1,0) = 0\} = (1-x, y)$ which contracts to $\mathfrak{q}$.

On the other hand, $\phi^{-1}(Q \oplus \mathbb{C}) = \{f \in R | f(0,y) \in Q\} \ni x$. But x does not belong to the ideal $(1-x, y)$, contradiction.

Hence the extension $R \hookrightarrow R'$ is an integral extension of integral domains but it does not satisfy going down property. However, we should not panic as R is not normal.

Problem 3.

Lemma 0.1. *Given $l \in \mathbb{N}$ and a finite number of pairwise different collections $(k_{1i}, \cdots, k_{l,i})$, $i = 1, \cdots, N$ There is a collection $b_1, \cdots b_l \in \mathbb{N}$ of positive natural numbers such that all the sums*

$$\sum_{j=1}^{l} b_j k_{ji}, i = 1 \cdots N$$

are pairwise different.

Proof. The equation $\sum\limits_{j=1}^{l} b_j(k_{ji} - k_{jr}) = 0$ defines a hyperplane in $\mathbb{R}^l$ and the complement to the union of these hyperplanes for $i \neq r$ is open and dense in $\mathbb{R}^l$. As $(\mathbb{Q}_{>0})^l$ is dense in $(\mathbb{R}_{>0})^l$ which is open in $\mathbb{R}^l$, the intersection of $(\mathbb{Q}_{>0})^l$ with the complement to the union of these hyperplanes is non-empty. In other words there is a collection $b_1, \cdots b_l \in \mathbb{Q}$ of positive rational numbers, such that all the sums

$$\sum_{j=1}^{l} b_j k_{ji}, i = 1 \cdots N$$

are pairwise different. Multiplying by the common denominator yields the desired collection of positive natural numbers.

For a more elementary proof one could use induction on l. $\square$

Lemma 0.2. *Given $l \in \mathbb{N}$, a finite number of pairwise different collections $(k_{1i}, \cdots, k_{l,i})$, $i = 1, \cdots, N$ and a collection $d_1, \cdots d_N \in \mathbb{N}$ There is a collection $c_1, \cdots c_l \in \mathbb{N}$ of positive natural numbers such that all the sums*

$$\sum_{j=1}^{l} c_j k_{ji} + d_i; i = 1, \cdots, N$$

are pairwise different.

Proof. Take the collection $b_1, \cdots b_l \in \mathbb{N}$ of positive natural numbers such that all the sums

$$\sum_{j=1}^{l} b_j k_{ji}, i = 1, \cdots, N$$

are pairwise different provided by Lemma 0.1 and multiply them by $\max\{d_1, \cdots d_N\} + 1$. We obtain a collection $c_1, \cdots c_l \in \mathbb{N}$ of positive natural numbers such that all the sums

$$\sum_{j=1}^{l} c_j k_{ji}, i = 1, \cdots, N$$

are pairwise different and differ by more than $\max\{d_1, \cdots d_N\}$. Thus all the sums

$$\sum_{j=1}^{l} c_j k_{ji} + d_i, i = 1, \cdots, N$$

are pairwise different.

Alternatively take $M := \max\{k_{j,i}, d_k\} + 1$ and set $c_i := M^{i+1}$. Then all the sums

$$\sum_{j=1}^{l} c_j k_{ji} + d_i; i = 1, \cdots, N$$

are pairwise different as they have pairwise different expansions in base M. □

Now write f as a sum

$$\sum c_{k_{1,i}, \cdots, k_{n-1,i}} x_1^{k_{1,i}} x_2^{k_{2,i}} \cdots x_{n-1}^{k_{n-1,i}} p_{k_{1,i}, \cdots, k_{n-1,i}}(x_n)$$

with pairwise different collections $(k_{1,i}, \cdots, k_{n-1,i})$. Observe that

$$(x_1 + x_n^{a_1})^{k_{1,i}} (x_2 + x_n^{a_2})^{k_{2,i}} \cdots (x_{n-1} + x_n^{a_{n-1}})^{k_{n-1,i}} p_{k_{1,i}, \cdots, k_{n-1,i}}(x_n)$$

is a polynomial monic in x_n of degree

$$\sum_{j=1}^{n-1} a_j k_{ji} + \deg(p_{k_{1,i},\cdots,k_{n-1,i}})$$

in x_n up to a scalar.

By Lemma 0.2 we can choose $a_1, \cdots, a_{n-1}$ so that all the above expressions are pairwise different. Therefore, $f(x_1 + x_n^{a_1}, \cdots, x_{n-1} + x_n^{a_{n-1}}, x_n)$ is monic in x_n up to a scalar and the assertion follows.