Problem 1a.

We can write every element v of M as $\overline{\frac{k}{p^n}}$ where k is coprime to p. Now let $M' \subset M$ be a submodule. Note that if $\overline{\frac{k}{p^n}} \in M$ for k coprime to p then $\overline{\frac{1}{p^n}} \in M$. Indeed, there are integers $r, s \in \mathbb{Z}$ such that $rk + p^n s = 1$, so

$$r\overline{\frac{k}{p^n}} = \overline{\frac{1}{p^n}} \in M.$$

Hence if $\overline{\frac{k}{p^n}} \in M'$ for k coprime to p then $\langle\overline{\frac{1}{p^n}}\rangle \subset M'$.

It follows that if there exists the largest power p^N of p appearing among all elements of M' in the presentation $\overline{\frac{k}{p^n}}$, where k is coprime to p, then

$$M' = \langle\overline{\frac{1}{p^N}}\rangle.$$

Otherwise

$$M' \supset \bigcup_i \langle\overline{\frac{1}{p^i}}\rangle = M,$$

hence $M' = M$ is not proper.

Problem 1b.

We have

$$\{0\} \subsetneqq \langle\overline{\frac{1}{p}}\rangle \subsetneqq \cdots \subsetneqq \langle\overline{\frac{1}{p^N}}\rangle \cdots$$

providing an infinite ascending non-stabilizing chain of M-submodules implying that M is not Noetherian.

On the other hand, suppose there is a descending chain

$$M \supsetneqq M_0 \supsetneqq \cdots$$

of M-submodules. Then M_0 is proper, hence it equals $\langle\overline{\frac{1}{p^N}}\rangle$, in particular it is finite as a group. So there can be only finite number of subgroups of M_0 and the descending chain of these should necessarily stabilize. Thus M is Artinian.

Problem 2.

Recall the following lemma (see Lemma 7.7.a in Gathmann's textbook):

Lemma 0.1. *Given a short exact sequence*

$$N \hookrightarrow M \twoheadrightarrow P$$

of R-modules. Then M is Noetherian iff N and P are.

As R is Noetherian, it is Noetherian as a module over itself. It follows by induction considering (split) short exact sequences

$$0 \to R \to R^{\oplus n+1} \to R^{\oplus n} \to 0$$

that any free module on finite number of generators is Noetherian.

Now suppose M is finite over R, so there is a surjection $R^{\otimes m} \twoheadrightarrow M$ for some $m \in \mathbb{N}$. As $R^{\oplus m}$ is Noetherian it follows that M is Noetherian.

If M is not finite over R, pick an element $v_1 \in M$. Inductively on n for $v_1, \cdots, v_n \in M$ pick an element $v_{n+1} \in M \setminus \langle v_1, \cdots, v_n \rangle$ which is possible by assumption. Setting $N_i \coloneqq \langle v_1, \cdots, v_n \rangle$ we have an infinite ascending chain

$$N_1 \subsetneqq N_2 \subsetneqq \cdots$$

of M-submodules, hence M is not Noetherian.

Problem 3.

As M is Noetherian, it follows by similar inductive argument as in Problem 2 that $M^{\oplus n}$ is Noetherian for any $m \in \mathbb{N}$. As N is finite, there is a surjection $R^{\oplus m} \twoheadrightarrow N$ for some $m \in \mathbb{N}$.

Tensoring the above surjection by M and using right-exactness of $M \otimes_R -$ we obtain a surjection $M \otimes_R R^{\oplus m} \twoheadrightarrow M \otimes_R N$.

It remains to note that $M \otimes_R R^{\oplus m} \simeq M^{\oplus m}$, hence it is Noetherian. Thus $M \otimes_R N$ is Noetherian as well by Lemma 0.1.