Recall the following:

Proposition 1. *For any ideal $I \trianglelefteq R$ we have*

$$\bigcap_{I \subset \mathfrak{p}} \mathfrak{p} = \sqrt{I}.$$

In particular if $V(I_1) \subset V(I_2)$ then $\sqrt{I_2} \subset \sqrt{I_1}$ and there is an inclusion-reversing bijection between radical ideals $\sqrt{I} = I \trianglelefteq R$ and Zariski closed subsets of $\operatorname{Spec}(R)$ *given by*

$$I \mapsto V(I); Z \mapsto \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Problem 4.

As $\operatorname{Spec} R \simeq \operatorname{Spec}(R/\sqrt{0})$ and primeness of $\sqrt{0}$ is equivalent to the quotient ring being a domain we are reduced to the following statement:

Lemma 1. *Given a reduced ring S. Then* $\operatorname{Spec}(S)$ *is irreducible iff S is a domain.*

Proof. If $\operatorname{Spec}(S)$ is irreducible but there are some nonzero elements $a, b \in S$ such that $ab = 0$, then

$$\operatorname{Spec}(S) = V((a)) \cup V((b)),$$

Whence either $V((a))$ or $V((b))$ is equal to $\operatorname{Spec}(S)$ by the irreducibility of the latter. Without loss of generality assume $V((a)) = \operatorname{Spec}(S) = V(0)$, then Proposition 1 implies that $a \in \sqrt{(0)} = (0)$, contradiction.

If S is a domain but $\operatorname{Spec}(S) = V(I_1) \cup V(I_2)$ for some proper closed $V(I_1)$ and $V(I_2)$ of $\operatorname{Spec}(S)$ then the prime ideal (0) is contained in some $V(I_j)$, hence $(0) \supset I_j$ implying that $I_j = (0)$ thus $V(I_j) = \operatorname{Spec}(R)$, contradiction. □

Problem 1.

Let $Z = V(I) \subset \operatorname{Spec}(R)$ be a closed subset. Then $Z \simeq \operatorname{Spec}(R/I)$ which is irreducible iff $\sqrt{0}$ is prime in R/I by Problem 4. As the preimage of $\sqrt{0} \trianglelefteq R/I$ is $\sqrt{I}$ we conclude that irreducibility of Z is equivalent to primeness of $\sqrt{I}$. Hence $V(\mathfrak{p})$ is irreducible. And If $V(I) = Z$ is irreducible then $\sqrt{I}$ is prime and $Z = V(\sqrt{I})$.

Problem 2.

Consider the short exact sequence

$$0 \to N \to M \to M/N \to 0.$$

For a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ apply the localisation at $\mathfrak{p}$ and use exactness of the localisation functor. We obtain a short exact sequence

$$0 \to N_{\mathfrak{p}} \to M_{\mathfrak{p}} \to (M/N)_{\mathfrak{p}} \to 0.$$

Thus $M_{\mathfrak{p}} \neq 0$ iff $N_{\mathfrak{p}} \neq 0$ or $(M/N)_{\mathfrak{p}} \neq 0$, which is equivalent to $\mathrm{Supp}\,(M) = \mathrm{Supp}\,(N) \cup \mathrm{Supp}\,(M/N)$.

Problem 3.

Suppose R is Noetherian and given an ascending chain of open subsets

$$U_0 \subset U_1 \subset \cdots \subset \mathrm{Spec}(R).$$

Denoting $Z_i := \mathrm{Spec}(R) \setminus U_i$ we obtain a descending chain of closed subsets

$$Z_0 \supset Z_1 \supset \cdots .$$

Denote by $I_i \trianglelefteq R$ the radical ideal corresponding to Z_i. Then by Proposition 1 we have an ascending chain of ideals of R

$$I_0 \subset I_1 \subset \cdots ,$$

Which stabilizes by our hypothesis, yielding stabilization of the initial chain of open subsets.

The converse is not true: consider the ring

$$R := \mathbb{C}[x_1, x_2, \cdots]/(x_1^2, x_2^2, \cdots).$$

As the ideal $I := (x_1, x_2, \cdots) \trianglelefteq R$ consists of nilpotent elements we have

$$\mathrm{Spec}(R) \simeq \mathrm{Spec}(R/I) = \mathrm{Spec}(\mathbb{C}),$$

and $\mathrm{Spec}(\mathbb{C})$ consists of one point and hence is a Noetherian topological space. But R is not Noetherian as I is not finitely generated.