Problem 2a.

As in the proof of Krull Principle Ideal Theorem we can take quotient by P_0 and localize at P_n and assume that R is a local integral domain with the maximal ideal P_n. We will prove the assertion by induction on n starting with $n = 2$. Base: given $0 \subsetneq P_1 \subsetneq P_2$ and $a \in P_2$, in particular $\operatorname{codim}(P_2) \geq 2$. If $a = 0$ we can set $P_1' := P_1$. Otherwise consider P_1 to be the minimal prime lying over a. We have $P_1' \neq (0)$ and by Krull Principle Ideal Theorem $\operatorname{codim}(P_1') \leq 1$ so

$$0 \subsetneq P_1' \subsetneq P_2.$$

Step: let $n > 2$ and suppose the assertion is true for $n-1$. Consider the chain $P_{n-2} \subsetneq P_{n-1} \subseteq P_n$. By the induction step there is a prime P_{n-1}' such that $P_{n-2} \subsetneq P_{n-1}' \subsetneq P_n$ and $a \in P_{n-1}'$. Now apply the inductive assumption to the chain $P_0 \subsetneq \cdots \subsetneq P_{n-1}'$.

Problem 2b.

Take any chain in an integral Noetherian R starting with $P_0 = (0)$ and choose a nonzero $a \in P_n$. Then of course $a \notin P_0$. For a concrete example consider $0 \subsetneq P_1 = (2) \trianglelefteq \mathbb{Z}$ and $a = 2 \in (2)$.

Problem 1.

See Gathmann, Corollary 11.17.

Problem 3a.

Let $\mathfrak{m} \trianglelefteq \mathbb{Z}[x]$ be a maximal ideal.

Lemma 1. *$\mathfrak{m} \cap \mathbb{Z} = (p)$ for some prime $p \in \mathbb{Z}$*

Proof. Indeed, $\mathfrak{m} \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ so it is either (p) or (0). So suppose it is (0). Then the ideal $\mathfrak{m}\mathbb{Q}[x] \trianglelefteq \mathbb{Q}[x]$ is proper and hence is generated by a polynomial $f(x)$ of positive degree. Without loss of generality we may assume $f \in \mathbb{Z}[x]$ and the greatest common divisor of all coefficients of f is 1.

We claim that $f \in \mathfrak{m}$. Indeed, $f \cdot n \in \mathfrak{m}$ for some non-zero integer n. But $\mathfrak{m}$ is prime and does not contain any non-zero integers by assumption, it follows that $f \in \mathfrak{m}$.

Next we claim that $\mathfrak{m} = (f)$. Indeed, take any $g \in \mathfrak{m}$. We know that $f|g$ in $\mathbb{Q}[x]$ so we can write $g = hf$ for $h \in \mathbb{Q}[x]$. But as the greatest common divisor of all coefficients of f is 1 it follows from Gauss' Lemma that $h \in \mathbb{Z}[x]$ and hence $\mathfrak{m} = (f)$.

But then we claim that $\mathbb{Z}[x]/(f)$ cannot be a field. Indeed, as $(f) \cap \mathbb{Z} = (0)$ the composite $\mathbb{Z} \to \mathbb{Z}[x] \to \mathbb{Z}[x]/(f)$ is injective. Now pick a large enough integer A such that $f(A) \notin \{0, \pm 1\}$. Then $f(A)$ has no inverse in $\mathbb{Z}[x]/(f)$ since otherwise there would exist $h(x) \in \mathbb{Z}[x]$ such that $f(x)|f(A) \cdot h(x) - 1$. But substituting $x = A$ yields $f(A)|f(A) \cdot h(A) - 1$ which is a contradiction. □

So $\mathfrak{m} \cap \mathbb{Z} = (p)$ by Lemma. Thus the morphism $\mathbb{Z}[x] \to \mathbb{Z}[x]/\mathfrak{m}$ factors through $\mathbb{Z}[x] \to \mathbb{Z}[x]/(p) = \mathbb{F}_p[x]$ and $\mathfrak{m}$ is a preimage of a maximal ideal $\mathfrak{p}$ of $\mathbb{F}_p[x]$.

We have $\mathfrak{p} = (g)$ for an irreducible $g \in \mathbb{F}_p[x]$, hence $\mathfrak{m} = (p, \tilde{g})$ where $\tilde{g} \in \mathbb{Z}[x]$ is a lift of g.

Problem 3b.

Let $(0) = P_0 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq P_n = \mathfrak{m} = (p, f)$ be a maximal chain of primes in $\mathbb{Z}[x]$, where $\mathfrak{m}$ is a maximal ideal written in the form provided by Problem 3a. As $p \in \mathfrak{m}$ by Problem 2a we may assume that $p \in P_1$. Then $P_1 \subsetneq P_2 \cdots \subsetneq P_n$ is the preimage of a chain of prime ideals $Q_1 \subsetneq Q_2 \cdots \subsetneq P_n$ in $\mathbb{F}_p[x]$ which should have length ≤ 1, hence $n \leq 2$. We proved that $\dim(\mathbb{Z}[x]) \leq 2$. For an example of length 2 chain consider $0 \subsetneq (p) \subsetneq (p, x)$.

Problem 4a.

Set $k = \dim(X)$ and $l = \dim(Y)$. It follows that there are finite injective homomorphisms

$$\phi_1 \colon K[x_1, \cdots, x_k] \hookrightarrow K[X]; \ \phi_2 \colon K[y_1, \cdots, y_l] \hookrightarrow K[Y].$$

Thus

$$\phi_1 \otimes \phi_2 \colon K[z_1 \cdots z_{k+l}] \simeq K[x_1, \cdots, x_k] \otimes K[y_1, \cdots, y_l] \hookrightarrow K[X] \otimes K[Y] = K[X \times Y]$$

is a finite injective homomorphism implying that $\dim(X \times Y) = k + l$.

Problem 4b.

Here we need to assume that every irreducible component of X and of Y has dimension $\dim(X)$ and $\dim(Y)$ respectively.

Suppose first that $Y = V(x_1) \simeq \mathbb{A}^{n-1}$, a coordinate hyperplane. We need to show that dimensions of irreducible components of $X \cap Y$ are at least $\dim(X) - 1$. We have $K[X] \simeq K[x_1, \cdots, x_n]/I_X$ and the irreducible components of $X \cap Y$ correspond to minimal primes of $K[x_1, \cdots, x_n]/(I_X, x_1)$, i.e. to minimal primes of $K[X]$ containing the image of x_1. By Krull Principle Ideal Theorem all these primes are of height at most 1, i.e. the corresponding subvarieties of $K[X]$ have dimension $\geq \dim(X) - 1$.

Now suppose $Y = V(x_1, \cdots, x_k) = V(x_1) \cap V(x_2) \cap \cdots \cap V(x_k) \simeq \mathbb{A}^{n-k}$. We have $X \cap Y = (X \cap (V(x_1) \cap \cdots \cap V(x_{k-1}))) \cap V(x_k)$ and it is easy to see by induction using the case $k = 1$ as the base that every irreducible component of this intersection has dimension at least $\dim(X) - k$.

Now take general X, Y. We have an embedding $X \times Y \hookrightarrow \mathbb{A}^{2n} = \mathbb{A}^n \times \mathbb{A}^n$ and $X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^{2n}}$, where $\Delta_{\mathbb{A}^{2n}}$ is the diagonal subvariety of $\mathbb{A}^n \times \mathbb{A}^n$ which is defined by the ideal $(x_1 - y_1, \cdots, x_n - y_n)$. As a corollary from Problem 4a every irreducible component of $X \times Y$ has dimension $\dim(X) + \dim(Y)$. Moreover, $\Delta_{\mathbb{A}^{2n}}$ is an $n-$dimensional intersection of n hyperplanes in $\mathbb{A}^{2n}$ and thus we reduce to the previous case and conclude that any irreducible component of $X \cap Y$ has dimension at least $\dim(X) + \dim(Y) + n - 2n = \dim(X) + \dim(Y) - n$.