Problem 1.

Denote $f = y^2 - x^3 - x^2$. We have

$$\frac{\partial f}{\partial x} = -3x^2 - 2x,\ \frac{\partial f}{\partial y} = 2y.$$

Let $\mathfrak{n} \trianglelefteq \mathbb{C}[x,y]/(f)$ be a maximal ideal. Then

$$V(\mathfrak{n}) = (a,b) \in \mathbb{C}^2$$

for some $(a,b) \coloneqq p \in \mathbb{C}^2$ and by Graham Construction 11.34 the tangent space at p is given by

$$T_pV(f) = V((-3a^2 - 2a)z + 2bt) \subset \mathbb{A}^2_{z,t}.$$

It is of dimension 1, unless $-3a^2 - 2a = 2b = 0$. As (a,b) moreover satisfy $f(a,b) = 0$ we can see that the latter holds iff $(a,b) = (0,0)$, i.e. $\mathfrak{n} = \mathfrak{m}$ and in this case $T_pV(f)$ is 2-dimensional. As tangent space is dual to cotangent their dimensions coincide.

As $\mathbb{C}[x,y]$ is an integral domain, so in particular f is not a zero divisor, any minimal prime over f has height 1, so $\dim V(f) = 2 - 1 = 1$. (Actually the only minimal prime over f is (f) as f is an irreducible element in the UFD $\mathbb{C}[x,y]$.)

Recall the following theorem proven on the lectures:

Theorem 1. *Assume an R-module M has a decomposition series. Then all decomposition series have the same length and any finite chain in M can be extended to a decomposition series.*

Problem 2.

Assume M is of finite length. Suppose there is an infinite strictly ascending or strictly descending chain of submodules of M. Take a piece of length $l(M)+1$ and extend it to a decomposition series. The obtained decomposition series has length strictly bigger than $l(M)$, so we obtain a contradiction.

Assume M is nonzero Artinian and Noetherian. By the descending chain condition there is a minimal nonzero submodule M_0 of M. If $M_0 \neq M$ similarly there is a minimal submodule M_1 of M properly containing M_0 and so on. Constructing on each step a minimal submodule containing properly previous ones leads to an increasing chain of submodules of M which should be finite by our hypothesis on M. Thus we obtain

$$0 \subsetneq M_0 \subsetneq \cdots \subsetneq M_{l-1} \subsetneq M_l = M.$$

By minimality conditions M_{i+1}/M_i are simple, so it is a decomposition series.

Problem 3.

We will prove the assertion by induction on $l = l(M)$. The base will be $l = 1, 2$. If $l = 1$ the statement is clear. Let $l = 2$ and suppose there are two different decomposition series $0 \subsetneq N \subsetneq M$ and $0 \subsetneq N' \subsetneq M$. Note that we have $N + N' = M$ as $N + N'$ is a submodule of M properly containing N.

Lemma 1. *$\phi\colon N \oplus N' \to N + N' = M$ is an isomorphism.*

Proof. Indeed, ϕ is surjective and its kernel is $N \cap N' = 0$. □

Now as $\phi|_{(N,0)}\colon N \hookrightarrow M$ is the embedding arising from the first decomposition series we have

$$M/N \simeq (N \oplus N')/(N, 0) \simeq N'$$

and similarly

$$M/N' \simeq (N \oplus N')/(0, N') \simeq N.$$

So the case $l = 2$ is proven.

Let $l \geq 3$ and assume the statement is true for $l-1$. Take two decomposition series

$$0 \subsetneq M_0 \subsetneq \cdots \subsetneq M_l = M$$

and

$$0 \subsetneq M'_0 \subsetneq \cdots \subsetneq M'_l = M.$$

If $M_0 = M'_0$ it follows that first successive quotients coincide and $0 \subsetneq M_1/M_0 \cdots \subset M_l/M_0$ as well as $0 \subsetneq M'_1/M_0 \cdots \subset M'_l/M_0$ are decomposition series for a length $l-1$ module M/M_0 so we conclude by inductive assumption.

If $M_0 \neq M'_0$ consider the chain $0 \subsetneq M_0 \subsetneq M_0 + M'_0$. It extends to a decomposition series of the form

$$0 \subsetneq M_0 \subsetneq M_0 + M'_0 \subsetneq \cdots \subsetneq M''_l = M.$$

Consider the extending of the chain $0 \subsetneq M'_0 \subsetneq M_0 + M'_0$ to the decomposition series

$$0 \subsetneq M'_0 \subsetneq M_0 + M'_0 \subsetneq \cdots \subsetneq M''_l = M$$

with the same higher terms as in the previous one starting with $M_0 + M'_0$.

Now observe that the first and the third decomposition series have the common first term, hence the same successive quotients by induction hypothesis. Similarly the second and the fourth decomposition series have the same successive quotients. Also the third and the fourth decomposition series have the same successive quotients by the case $l = 2$ and as they have the same terms starting with $M_0 + M'_0$. Therefore the first and the second decomposition series have the same successive quotients.