Practice problems
Commutative Algebra: Fall 2023
Exam

Problem 1. Let R be a ring, I an ideal. Assume R/I is flat. Show $I = I^2$.

Problem 2. Let $\varphi : R \hookrightarrow S$ be an integral extension. Show that the induced map $\varphi^* : \mathrm{Spec} S \to \mathrm{Spec} R$ is closed (that is, closed sets map to closed sets).

Problem 3. Let $R \subset S$ be an integral extension, $\mathfrak{p}$ prime in R. Let $\mathfrak{q}$ be a prime lying over $\mathfrak{p}$. Show that $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.

Problem 4. Let $R \subset S$ be an integral extension, $\mathfrak{p}$ a prime in R, and suppose that there is a unique prime $\mathfrak{q}$ in S lying over $\mathfrak{p}$. Show that $\mathfrak{q}S_{\mathfrak{p}}$ is the maximal ideal of $S_{\mathfrak{p}}$. Show that $S_{\mathfrak{q}} = S_{\mathfrak{p}}$. Show that $S_{\mathfrak{q}}$ is integral over $R_{\mathfrak{p}}$.

Problem 5. Show that $\{f \in \mathbb{Q}[t] : f(0) \in \mathbb{Z}\}$ is a ring that is not Noetherian.

Problem 6. Let A be the ring of functions $f : \mathbb{C} \to \mathbb{C}$ that are holomorphic in a neighborhood of 0 (sometimes this is called the ring of formal power series $\mathbb{C}[[x]]$). Show that A is a discrete valuation ring.

Problem 7. Let R be a valuation ring and K its field of fractions. Show that every ring A such that $R \subset A \subset K$ is the localization of R at a prime ideal.