LECTURE 2 EXERCISES

Exercise 1. In this exercise, we start with a vector space $N_{\mathbb{R}}$ and an inner product $\langle , \rangle$ $N_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. This induces a metric on $N_{\mathbb{R}}$ as usual by

$$d(x, y) = \sqrt{\langle y - x, y - x \rangle}$$

- Let σ be a cone in $N_{\mathbb{R}}$, and $x \notin \sigma$. Show that there is a unique point $y \in \sigma$ closest to x (hint: if there are two points y_1, y_2 in σ equidistant to x, show that the midpoint $\frac{y_1+y_2}{2}$ is in σ and is closer to x).
- Let y be the point of part (i). Show that
$$\langle x, y \rangle = \langle y, y \rangle$$
- Show that the vector $u = y - x$ is perpendicular to y and negative on x. (Hint: there is a right triangle formed by y, x and u).

Exercise 2. Prove the characterization of faces from the lecture: $\tau \subset \sigma$ is a face of σ if and only if τ is a cone, and $x + y \in \tau \implies x \in \tau$ and $y \in \tau$.

(Hint: Let $\overline{\tau}$ be the smallest face containing τ. Then, since $\overline{\tau}$ is the disjoint union of its interior and its proper faces, there must be a point $x \in \tau \cap \overline{\tau}^{\circ}$. Use the fact that x is interior to show that for any $y \in \overline{\tau}$, $ax - y \in ((\overline{\tau})^{\vee})^{\vee} = \overline{\tau}$ for $a >> 0$, and use and the assumption.

Exercise 3. Let σ be a cone, and τ a face. Suppose ρ is a face of τ. Show that ρ is a face of σ.

(Hint: what this says is: if $\rho = u^{\perp} \cap \tau$, for $u \geqslant 0$ on τ, it is also $w^{\perp} \cap \sigma$, for $w \geqslant 0$ on σ. Use the description $\tau = v^{\perp} \cap \sigma$, for $v \geqslant 0$ on σ to construct such a w from u.)

(Hint 2: for an easier proof, use exercise 2).

Exercise 4. Show that every proper face τ in σ is contained in a facet.

(Hint: Look at the image of τ and σ in the quotient space $N_{\mathbb{R}}/\mathrm{Span}(\tau)$).

Exercise 5.

Definition 1. Let A be an integral domain, with field of fractions K. An element x of K is integral over A if it satisfies a (observe: *monic*) equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots a_n = 0$$

with $a_i \in A$. The ring A is called normal if every integral element $x \in K$ is in A.

- Show that the intersection of normal domains A_i in K is normal.
- Let σ be a cone, and let ρ be a ray of σ (one dimensional face). Compute
$$\mathbb{C}[S_\rho]$$
and show that it is normal.
- Show that $\mathbb{C}[S_\sigma]$ is normal. (Hint: Observe that $\sigma^{\vee} = \cap \rho^{\vee}$, as ρ ranges through the rays of σ.)

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Exercise 6. Suppose $V(F, N)$ is a toric variety corresponding to the fan F in N, and $V(G, L)$ is the fan corresponding to G in L. Construct the lattice and the fan of $V(F, N) \times V(G, L)$.

Exercise 7. Suppose (σ, N), (τ, L) are cones in lattices, and $\phi : N \to L$ is a homomorphism of lattices such that the induced map $\phi_{\mathbb{R}} : N_{\mathbb{R}} \to L_{\mathbb{R}}$ takes σ into τ.

- Check that the homomorphism $\phi : N \to L$ induces a homomorphism of tori $\operatorname{Spec} \mathbb{C}[N^\star] \to \operatorname{Spec} \mathbb{C}[L^\star]$.
- Show that there is a natural map

$$V(\sigma, N) = \operatorname{Spec} \mathbb{C}[S_\sigma := \sigma^\vee \cap N^\star] \to V(\tau, L) = \operatorname{Spec} \mathbb{C}[S_\tau := \tau^\vee \cap L^\star]$$

- Check that the map $V(\sigma, N) \to V(\sigma, L)$ of part (2) is equivariant. (Hint: this boils down to the claim that the diagram

$$\begin{array}{ccc} \mathbb{C}[S_\tau] & \longrightarrow & \mathbb{C}[S_\tau] \otimes \mathbb{C}[L^\star] \\ \downarrow & & \downarrow \\ \mathbb{C}[S_\sigma] & \longrightarrow & \mathbb{C}[S_\sigma] \otimes \mathbb{C}[N^\star] \end{array}$$

commutes. It also boils down to the expectation that you constructed the correct map in (2) ;))

Exercise 8. Suppose $\sigma_1, \sigma_2, \sigma_3$ are cones in a fan F, and let $\tau_{12}, \tau_{13}, \tau_{23}$ be the intersections of $\sigma_i \cap \sigma_j$ $(i, j = 1, 2, 3)$, which are faces of each σ_i, σ_j. Let ρ be the intersection $\sigma_1 \cap \sigma_2 \cap \sigma_3$. Show that the composition

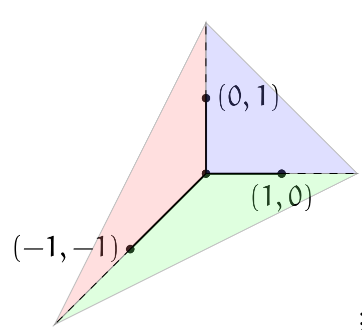
$$V(\rho, N) \to V(\tau_{12}) \to V(\sigma_1)$$

and

$$V(\rho, N) \to V(\tau_{13}) \to V(\sigma_1)$$

coincide.

Exercise 9. Find the variety corresponding to the following fans:



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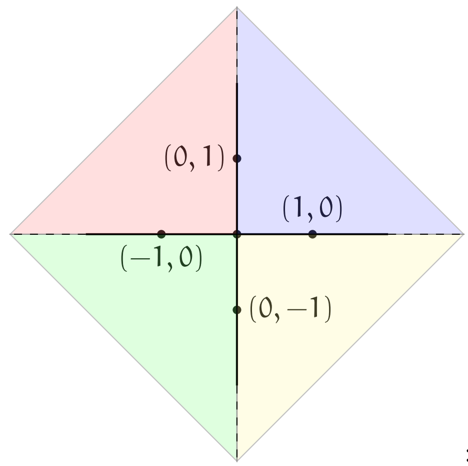
(0, 1)
(1, 0)
(−1, 0)
(0, −1)

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