LECTURE 5 EXERCISES

Exercise 1. Classify the smooth affine toric varieties.

For the following exercises, let $X = \operatorname{Spec} A$, $A = \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_l)$ be a variety over $\mathbb{C}$, $x \in X$ a point of X, corresponding to a maximal ideal m. A derivation D at x is a $\mathbb{C}$-linear map

$$D : A \to \mathbb{C}$$

that satisfies the Leibniz Rule

$$D(fg) = f(x)D(g) + g(x)D(f)$$

The tangent space $T_{X,x}$ is defined as the vector space of all derivations at x.

Exercise 2. Show that

$$T_{X,x} = \operatorname{Hom}(m/m^2, \mathbb{C})$$

Exercise 3. Let ϕ be the function on X defined by

$$\phi(x) = n - \text{rank} J(f_1, \cdots, f_l)$$

where $J(f_1, \cdots, f_l)$ is the Jacobian matrix

$$(\partial f_i / \partial x_j)$$

Show that the function ϕ is upper semicontinuous (this means that the set $\{x \in X : \phi(x) < C\}$ is open in the Zariski topology of X for any $C \in \mathbb{R}$).

Exercise 4. Let F be the fan with four maximal (three dimensional) cones generated by choosing three out of four of $e_1, e_2, e_3, -e_1 - e_2 - e_3$. Compute the fan of the star subdivision Star(ν) for the following ν:

- $\nu = e_1$
- $\nu = e_1 + e_2$,
- $\nu = e_1 + e_2 + e_3$

Sketch the algebraic variety corresponding to F and the above star subdivisions, using the orbit-cone correspondence. Are the varieties above singular or not?