The Orbit-Cone Correspondance

Jeremy Feusi

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1 Introduction

In this talk we will prove the so-called "orbit-cone" correspondence. As the name suggests, this will give a correspondence between the cones contained in the fan Δ and the orbits of the action of T on $X(\Delta)$. As a corollary, we have that the number of orbits is finite, which one would expect in any case since the torus acts transitively on itself and hence the dense open torus $T \subseteq X$ is contained in a single orbit (in fact we will show that it is an orbit in itself). To give some motivation for what is to come, we start with some examples:

Example 1.1. *In this example, we consider the toric variety $\mathbb{C}^2$, corresponding to the cone Δ spanned by e_1 and e_2 in $\mathbb{Z}^2$. This variety is visualized in Figure 1. Since $(\mathbb{C}^*)^2$ acts by componentwise multiplication, it is easy to see that the orbits are exactly the sets highlighted in Figure 1. Observe that the closure of any orbit is a union of orbits and the relationship between the orbits with respect to the*

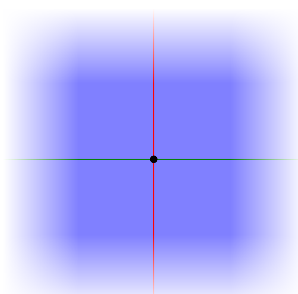


Figure 1: Visualization of $\mathbb{C}^2$, with different orbits highlighted by using different colors

partial order $X \subseteq \overline{Y}$, can be visualized as follows:

$$\begin{tikzcd}
 & (\mathbb{C}^*)^2 \arrow[dl] \arrow[dr] & \\
\mathbb{C}^* \times 0 \arrow[dr] & & 0 \times \mathbb{C}^* \arrow[dl] \\
 & \{(0,0)\}. &
\end{tikzcd}$$

So far so good, but no interesting patterns are immediately visible from a single example. Let's try the same thing with a singular surface example:

Example 1.2. *Let $X := Z(xy - z^2)$ be the singular cone. This is a toric variety corresponding to the fan spanned by e_1 and $2e_2 + e_1$ in $\mathbb{Z}^2$. Indeed, the dense open torus in X is given by $D(z)$. Namely, $z \neq 0$ implies $x, y \neq 0$, so on $D(z)$ we may invert x and $xy - z^2 = 0$ becomes equivalent to $y = \frac{z^2}{x}$. Hence*

$$D(z) \cong Z[x, z]_{x,z} \cong (\mathbb{C}^*)^2.$$

The torus $D(z)$ acts on itself by componentwise multiplication $(\alpha, \beta) \cdot (x, z) = (\alpha x, \beta z)$ and extending this action to x means that we must send y to $\frac{\beta^2}{a} y$. Hence, we obtain the action:

$$(\alpha, \beta) \cdot (x, y, z) = \left(\alpha x, \frac{\beta^2}{\alpha} y, \beta z\right).$$

From this description, we can immediately determine the orbits, which are visualized below, in the same way as for Example 1.1.

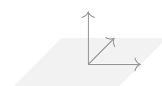
$$\begin{tikzcd}
 & X \cap (\mathbb{C}^*)^2 \arrow[dl] \arrow[dr] & \\
\mathbb{C}^* \times 0 \times 0 \arrow[dr] & & 0 \times \mathbb{C}^* \times 0 \arrow[dl] \\
 & \{(0,0,0)\}. &
\end{tikzcd}$$

This is already more interesting: The structure of the orbits in Example 1.1 is identical to the structure in Example 1.2 and again, we see that the closure of the orbits is a union of orbits. In light of what we are hoping to show however, this does make sense, since the combinatorial structure of Δ in the two cases is also identical. Let us try a higher dimensional example:

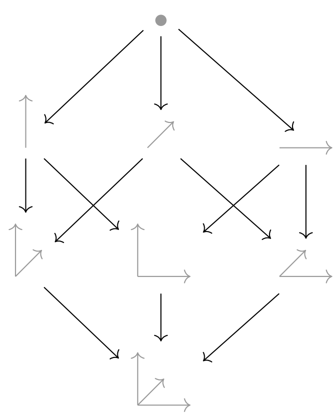
Example 1.3. *Consider the toric variety $\mathbb{C}^n$ with the action of $(\mathbb{C}^*)^n$ given by componentwise multiplication. Clearly, the orbits are in 1-to-1 correspondence* *with the subsets of* $\{1, \ldots, n\}$ *where* $\{i_1, \ldots, i_j\}$ *corresponds to the set of points* $z \in \mathbb{C}^n$ *such that* $z_{i_j} = 0$ *for all* j*. For example, in* $\mathbb{C}^3$*, we have the structure:*

$$\begin{tikzcd}
& \{\} = (\mathbb{C}^*)^3 \arrow[dl] \arrow[d] \arrow[dr] & \\
\{1\} \arrow[d] \arrow[dr] & \{2\} \arrow[dl] \arrow[dr] & \{3\} \arrow[dl] \arrow[d] \\
\{1,2\} \arrow[dr] & \{1,3\} \arrow[d] & \{2,3\} \arrow[dl] \\
& \{1,2,3\} = \{0\}. &
\end{tikzcd}$$

Note that $\mathbb{C}^3$ *corresponds to the fan* Δ *generated by* e_1*,* e_2 *and* e_3 *in* $\mathbb{Z}^3$*, visualized as:*



Observe that the cones contained in Δ *have a structure with respect to inclusion which can be visualized as:*



Ok, nice! Let's see what happens if we work out an example with multiple cones:
Example 1.4. *Consider the fan corresponding to* $\mathbb{P}^2$*, pictured in Figure 2. In homogeneous coordinates, the torus action on* $\mathbb{P}^2$ *is given by*

$$(a, b) \cdot [z_0 : z_1 : z_2] = [az_0, bz_1, b^{-1}z_2].$$

Thus we clearly see that the orbits are given by $D(z_0z_1z_2)$ *as well as* $Z(z_0)$*,* $Z(z_1)$*,* $Z(z_2)$ *and their pair-wise intersections. Their structure is as follows:*

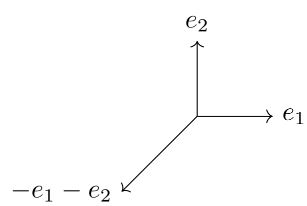
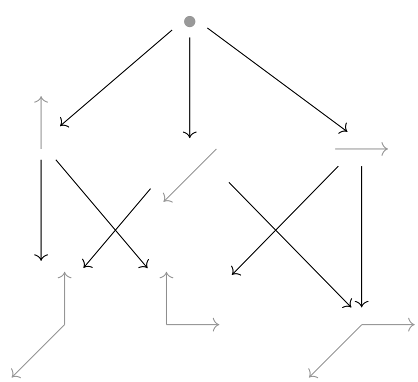


Figure 2: The fan corresponding to $\mathbb{P}^2$.

$$\begin{tikzcd}
& D(z_0 z_1 z_2) \arrow[dl] \arrow[d] \arrow[dr] & \\
Z(z_0) \arrow[d] \arrow[dr] & Z(z_1) \arrow[dl] \arrow[dr] & Z(z_2) \arrow[dl] \arrow[d] \\
Z(z_0) \cap Z(z_1) & Z(z_0) \cap Z(z_2) & Z(z_1) \cap Z(z_2).
\end{tikzcd}$$

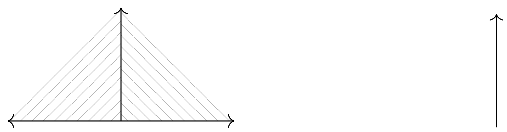
As in the preceding example, we can again look at the structure of the cones contained in Δ:



The above structure looks astoundingly similar to the structure of the orbits. In fact, this is not a coincidence but instead a first glimpse at the result we will now attempt to prove. The first question which naturally arises is whether bigger cones should correspond to bigger orbits or vice versa. To give a heuristic for this, observe that every toric variety has a unique dense orbit (the dense open torus) and that every fan has a unique cone of dimension 0 (the point at the origin). This suggests that these two objects should correspond to each other and hence smaller cones correspond to larger orbits. Indeed, the following theorem holds:

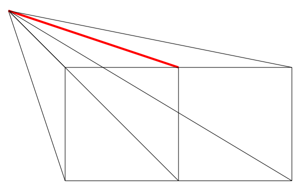
Theorem 1.1. *There exists an inclusion-reversing bijection between the cones τ in a fan Δ and orbits O_τ in $X(\Delta)$. Moreover, we have the following properties:*

1. *$V(\tau) := \overline{O}_\tau$ is again a toric variety with torus action restricted from $X(\Delta)$.*

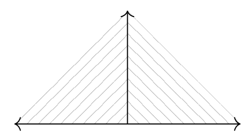


(a) Δ (b) The image of Δ in $N(\tau)$

Figure 3: The result of taking the image of Δ in $N(\tau)$,where τ is the cone spanned by e_1.



(a) Δ



(b) The image of in $N(\tau)$ of the cones containing τ.

Figure 4: The result of taking the image of the cones containing τ in $N(\tau)$,where τ is the cone marked in red.

2. *$dim(\tau) = codim(V(\tau))$.*
3. *$V(\tau) = \coprod_{\sigma \succ \tau} O_\sigma$.*
4. *$U_\sigma = \coprod_{\tau \prec \sigma} O_\tau$ for all cones $\sigma \in \Delta$.*

The proof and explanation of this theorem will keep us busy for most of the remaining part of the lecture. We begin by constructing the bijection. For this, let Δ be a fan and $\tau \in \Delta$ a cone. The idea is that τ corresponds to a toric variety $V(\tau) \subseteq X(\Delta)$, whose torus action is compatible with the torus action on $X(\Delta)$. Namely, assuming that we have constructed such a $V(\tau)$, then the dense open torus $O_\tau \subseteq V(\tau)$ will be the orbit to which τ corresponds. We now construct $V(\tau)$:

First observe that if τ is of dimension k, then $V(\tau)$ should be of dimension $n - k$ (the bijection is supposed to be inclusion reversing). Thus, intuitively, we expect that we must somehow "quotient by τ". To make this rigorous, let us first define the lattice $N_\tau := N \cap \langle\tau\rangle$, i.e. the part of the lattice N lying in the subspace spanned by τ. Then $N(\tau) := N/N_\tau$ is again a lattice with dual lattice $M(\tau) := M \cap \tau^\perp$, where $\tau^\perp := \{\sigma \in M : \sigma(\tau) = 0\}$. In general, taking the image of Δ in $N(\tau)$ will not be so well behaved. For example, in Figure 3, the two cones in Δ are mapped to the same cone in $N(\tau)$.

However, restricting to the cones containing τ, one can see that in fact we do obtain a 1-to-1 correspondence between the set of these cones and their images in $N(\tau)$ (see for example Figure 4).

We call the resulting fan $Star(\tau)$. The claim is then that we can choose $V(\tau) :=$

$X(Star(\tau))$. To be able to do this, we must describe a closed embedding $V(\tau) \hookrightarrow X(\Delta)$. We construct this embedding on the level of cones $\sigma \in \Delta$ and their images $\overline{\sigma} \in Star(\tau)$. Obvious compatibility conditions then allow us to glue these together to obtain the required embedding.

The algebro-geometric machinery now tells us that a closed embedding $U_{\overline{\sigma}} \hookrightarrow U_\sigma$ is equivalent to a surjective morphism:

$$\mathbb{C}[S_\sigma] \twoheadrightarrow \mathbb{C}[S_{\overline{\sigma}}].$$

This surjection will be induced by a surjection

$$M \cap \sigma^\vee = S_\sigma \twoheadrightarrow S_{\overline{\sigma}} = M(\tau) \cap \overline{\sigma}^\vee = M \cap \tau^\perp \cap \sigma^\vee.$$

Indeed, the right-hand side is precisely the set of $u \in M \cap \sigma^\vee$ such that $u|_\tau = 0$. Hence, we obtain a canonical surjection by sending χ^u to χ^u if $u|_\tau = 0$ and to 0 if not. As a remark that we will be using later on, observe that on the dual monoids, this morphism corresponds to the "extension by 0" of a morphism $S_{\overline{\sigma}} \to \mathbb{C}$ to a morphism $S_\sigma \to \mathbb{C}$. There is an obvious commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[S_\sigma] & \longrightarrow & \mathbb{C}[S_{\overline{\sigma}}] \\ \downarrow & & \downarrow \\ \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M] & \longrightarrow & \mathbb{C}[S_{\overline{\sigma}}] \otimes \mathbb{C}[M(\tau)] \end{array}$$

which translates to the fact that the inclusion $U_{\overline{\sigma}} \to U_\sigma$ is equivariant with respect to the torus action. In more down to earth terms, this means that the torus action on $U_{\overline{\sigma}}$ is the restriction of the torus action on U_σ. Gluing these inclusions, we get exactly the embedding $V(\tau) \to X(\Delta)$ as claimed. We denote the image of $T_{N(\tau)}$ in $X(\Delta)$ by O_τ. First, observe that since the torus action on O_τ is transitive, this set must be contained in some orbit of $X(\Delta)$. In fact, this set is the entire orbit as the following proposition shows:

Proposition 1.1. *Let σ be a cone in some lattice N. Then:*

a. $U_\sigma = \coprod_{\tau \prec \sigma} O_\tau$
b. $V(\tau) = \coprod_{\gamma \succ \tau} O_\gamma$
c. $O_\tau = V(\tau) \setminus \coprod_{\gamma \succneqq \tau} O_\gamma = V(\tau) \setminus \coprod_{\gamma \succneqq \tau} O_\gamma = V(\tau) \setminus \bigcup_{\gamma \succneqq \tau} V(\gamma)$

Proof. For (a), observe that points in U_σ correspond to $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S_\sigma] \to \mathbb{C}$ (think "evaluating $f \in \mathbb{C}[S_\sigma]$ at a point"). This in turn corresponds to a morphism of monoids $x : S_\sigma \to \mathbb{C}$. Observe that x corresponds to a point in T_N if and only if $x(u) \in \mathbb{C}^*$ for all $u \in S_\sigma$, since in this case, it must come from a morphism $S_\sigma^{gp} \to \mathbb{C}$. But now, since $x(u + u') = x(u) \cdot x(u')$, this quantity is in $\mathbb{C}^*$ if and only if $x(u)$ and $x(u')$ are both in $\mathbb{C}^*$. This means that $x^{-1}(\mathbb{C}^*)$ must be the dual of some face τ of σ ($x^{-1}(\mathbb{C}^*) = \sigma^\vee \cap \tau^\perp \cap M$) (any point in the interior of a cone can be written as a convex combination of vectors in the boundary). From the definition of $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\overline{\sigma}}]$ above, one then

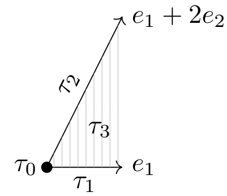


Figure 5: The cone corresponding to the singular cone discussed in Example 1.5.

sees that x can be written as the extension by 0 of some point $\overline{x}$ in $\mathbb{C}[S_{\overline{\sigma}}]$ and indeed $\overline{x}$ is contained in the image O_τ of the dense open torus $T_{N(\tau)}$. This shows (a). (b) follows from the description of $Star(\tau)$ as well as (a) and (c) follows from (b). $\square$

And with this, we have finally finished the proof of Theorem 1.1! We will end the talk in a minute, but let us just take a moment to see what we have achieved by applying it to a specific example (well-known by now):

Example 1.5. *We look again at the singular cone $X = Z(xy - z^2)$, discussed in Example 1.2. This toric variety comes from the cone depicted in Figure 5, by the correspondences $x = \chi^{e_2^*}$, $y = \chi^{e_1^*+2e_2^*}$ and $z = \chi^{e_1^*}$.*

Starting with the easy part, we clearly have that $V(\tau_0) = X$ and thus $O_\tau = T_N = D(z) \cong k[x, x^{-1}, z, z^{-1}]$. Next, looking at $V(\tau_1)$, we have $N(\tau_1) = \langle e_2 \rangle \cong \mathbb{Z}$ with the image of τ_3 in $N(\tau_1)$ equal to the cone spanned by $2e_2$. Thus $V(\tau_1) \cong \mathbb{A}^1$ and using the description of the surjection $\mathbb{C}[x, y, z] \to \mathbb{C}[t]$ described above, we have that it is defined by $(x, y, z) \mapsto (t, 0, 0)$. Hence the image of $V(\tau_1)$ in X is exactly $\mathbb{C} \times 0 \times 0$ and O_{τ_1} is $\mathbb{C}^* \times 0 \times 0$, which is precisely one of the orbits determined in example 1.2. Proceeding similarly, we find that $V(\tau_2) \cong \mathbb{A}^1$, corresponding to $0 \times \mathbb{C} \times 0 \subseteq X$ and $V(\tau_3) \cong \{*\}$, corresponding to $(0, 0, 0) \in X$.