One parameter groups

401-3140-73L Toric Geometry

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After we saw last week how to get a toric variety $X(\Delta)$ from a fan Δ we will now see how to recover a fan given a toric variety. Denote by $\mathbb{G}_m$ the multiplicative group of $\mathbb{C}^*$ seen as algebraic group, meaning as a variety with a group operation such that said group operation and inversion are regular maps. We get that

$$\mathrm{Hom}_{\text{alg.gr.}}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$

Without proof one way to see this is to note that such homomorphisms must look like Laurent polynomials from $\mathbb{C}^* \to \mathbb{C}^*$. Since all zeroes and singularities must be at 0 or ∞ it can only have one term and since $1 \mapsto 1$ the lead coefficient must be 1 these polynomials are of the form

$$z \mapsto z^k \text{ for } k \in \mathbb{Z}$$

Recall that if S is a semigroup then points of $X = \operatorname{Spec} \mathbb{C}[S]$ are maximal ideals of $\mathbb{C}[S]$ that correspond to semigroup morphisms $S \to \mathbb{C}$ witch are completely defined by the images of generators of S so if $x \in X$, meaning $x : S \mapsto \mathbb{C}$, then $\chi^s(x) = x(s)$. For example we have $\mathbb{C} \cong \operatorname{Spec} \mathbb{C}[X]$ where the morphism sending X to z is identified with the ideal $(X - z)$. For semigroups generated by cones in lattices N with dual lattice M we have $S_0 = M$ thus

$$\mathbb{T}_N \cong \operatorname{Spec} \mathbb{C}[S_0] \cong \mathrm{Hom}(M, \mathbb{C}^*)$$

Note that since x corresponds to $\varphi(s) = z^s(x)$ (evaluation at x), where z^s is multiindex notation for a Laurent monomial, $0 \in \mathbb{C}$ is not hit. We conclude that elements of the torus can be written as maps $M \to \mathbb{C}^*$.

One-parameter subgroups and characters

A one-parameter subgroup is a homomorphism $\mathbb{G}_m \to \mathbb{T}_N$ and since $\mathbb{T}_N \cong (\mathbb{C}^*)^n$ we get that the set of all one-parameter subgroup can be written as

$$\mathrm{Hom}(\mathbb{G}_m, \mathbb{T}_N) \cong \mathrm{Hom}(\mathbb{Z}, N) \cong N$$

Again without proof this can be seen by noting that if $\varphi \in \mathrm{Hom}(\mathbb{G}_m, \mathrm{Hom}(M, \mathbb{G}_m))$, then we get that $\varphi(-)(u) \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$, thus is dual to $u \in M$. So every one-parameter subgroup is given as

$$\lambda_v : \mathbb{C}^* \to \mathbb{T}_N \cong (\mathbb{C}^*)^n : z \mapsto (z^{v_1}, \ldots, z^{v_n})$$

for a $v \in N$. Dually we get for the set of all characters $\mathbb{T}_n \to \mathbb{G}_m$ that

$$\mathrm{Hom}(\mathbb{T}_N, \mathbb{G}_m) \cong \mathrm{Hom}(N, \mathbb{Z}) \cong M$$

This just means that $\chi^u : \mathbb{T}_n \to \mathbb{G}_m$ can be seen as a normed Laurent monomial

$$\chi^u(t) = {t_1}^{u_1} \cdots {t_n}^{u_n}$$

Since N and M are dual to each other we get a dual pairing $\mathrm{Hom}(\mathbb{T}_N, \mathbb{G}_m) \times \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}_N) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$, namely

$$\chi^u(\lambda_v(z)) = \chi^u(z^{v_1}, \ldots, z^{v_n}) = (z^{v_1})^{u_1} \cdots (z_{v_n})^{u_n} = z^{\langle u,v \rangle}$$

Also note that the identifications above essentially correspond to choosing a basis of the lattices formed by one-parameter subgroups and characters. So in conclusion we managed to described N and M intrinsically in terms of the torus giving us the identifications

$$\begin{array}{ccc} M \times N & \to & \mathbb{Z} \\ (u,v) & \mapsto & \langle u,v \rangle \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} \mathrm{Hom}(\mathbb{T}_N, \mathbb{G}_m) \times \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}_N) & \to & \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \\ (\chi^u, \lambda_v) & \mapsto & z \mapsto z^{\langle u,v \rangle} \end{array}$$

Limits and distinguished points

Now that we got N and M in terms of the torus we want to get a fan in N that generates the variety. Let $X(\Delta)$ be the variety generated by a fan Δ and let σ be a cone in Δ then define $U_\sigma = \operatorname{Spec}\mathbb{C}[S_\sigma]$ which is an affine variety in $X(\Delta)$. Note that for $\tau < \sigma$ (τ is a face of σ) we have $U\tau \subseteq U_\sigma$. For all cones σ such that $\tau < \sigma$ ($\tau = \sigma$ is possible) define the distinguished point x_τ of τ in $U\sigma$ as the point represented by the semigroup homomorphism

$$x_\tau : U_\sigma \to \mathbb{C} : u \mapsto \begin{cases} 1 & \text{if } u \in \tau^\perp \\ 0 & \text{else} \end{cases}$$

This is well defined since $\tau^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$. We now look at limits

$$\lim_{z\to 0} \lambda_v(z)$$

for various $v \in N$ and at their relations to the distinguished points described above. As a quick side note since our varieties are equipped with the Zariski topology the limit should be understood as a solution to the following extension problem

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "f"] \arrow[d, hook] & \mathbb{T}_N \arrow[d, hook] \\ \mathbb{C} \arrow[r, dashed, "\tilde{f}"] & X(\Delta) \end{tikzcd}$$

Where we say that if such a $\tilde{f}$ exists we set $\tilde{f}(0) = \lim_{z\to 0} f(z)$. As an example assume $e_1, \dots, e_k$ are part of a basis of N and generate the cone σ. Then $\sigma^\vee$ is generated by $e_1^*, \dots, e_k^*, e_{k+1}^*, -e_{k+1}^*, \dots, e_n^*, -e_n^*$ and thus

$$U_\sigma = \operatorname{Spec}\mathbb{C}[X_1, \dots, X_k, X_{K+1}, X_{k+1}^{-1}, \dots, X_n, X_n^{-1}] \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$$

$\lim_{z\to 0} \lambda_v(z) = (z^{v_1}, \dots, z^{v_n})$ therefore exists if $v_i = 0$ for $i \geq k+1$ and if $v_i \geq 0$ for $i \leq k$. Then the limit is given by $(\delta_1, \dots, \delta_k, 1 \dots, 1)$ with $\delta_i = 1$ for $v_i = 0$ and $\delta_i = 0$ if $v_i \geq 1$. Note that all these combinations of 0 and 1 for the δ_i correspond to the distinguished points of the faces of σ. If we look at the case $n = k = 1$ with $X(\Delta) = \operatorname{Spec}\mathbb{C}[X_1] \cong \mathbb{C}$ we get that $U_\sigma \cong \mathbb{C}$ in which case the limit for e_1 exists and the limit for $-e_1$ doesn't exist as z^{-1} has no limit in $\mathbb{C}$ but if we look at the fan generated by e_1 and $-e_1$ giving us $X(\Delta) \cong \mathbb{P}^1$ the limit suddenly exists as $[1 : z^{-1}] = [z : 1]$ as $\mathbb{P}^1$ is patched together from $\operatorname{Spec}\mathbb{C}[X_1]$ and $\operatorname{Spec}\mathbb{C}[X_1^{-1}]$ with the glueing map $x_1 \mapsto x_1^{-1}$. In this example we see that if the limit exists it's the distinguished point of the cone containing v and if it doesn't v isn't part of the fan. This gets formalized in the following two propositions.

Proposition 1. *If $v \in |\Delta|$ and τ is the cone with v in it's relative interior, then the limit exists and $\lim_{z\to 0} \lambda_v(z) = x_\tau$.*

Proof. We look at any U_σ with $\tau < \sigma$. Then $\lambda_v(z) : M \to \mathbb{C}^*$ is an element of the torus thus given by $u \mapsto z^{\langle u,v\rangle}$. $u \in S_\sigma = \sigma^\vee \cap M$ now implies that $\langle u, v\rangle \geq 0$ and equality holds if and only if $u \in \tau^\perp$, thus

$$\lim_{z\to 0} \lambda_v(z)(u) = \begin{cases} \lim_{z\to 0} z^0 = 1 & \text{if } u \in \tau^\perp \\ \lim_{z\to 0} z^{\langle u,v\rangle} = 0 & \text{else} \end{cases} = x_\tau$$

$\square$

Proposition 2. *If $v \notin |\Delta|$ then $\lim_{z\to 0} \lambda_v(z)$ doesn't exist in $X(\Delta)$.*

Proof. As $X(\Delta)$ is glued together by affine varieties that come from cones it suffices to show that $v \notin U_\sigma$ for all cones σ in the fan Δ. Since $(\sigma^\vee)^\vee = \sigma$ there exists a $u \in \sigma^\vee$ with $\langle u, v\rangle < 0$, then $z^{\langle u,v\rangle} \to \infty$ as $z \to 0$ thus doesn't exist in U_σ. $\square$

The second proposition especially implies that if $X(\Delta)$ is a full compactification of a torus the fan Δ is complete, meaning $|\Delta| = \mathbb{R}^n$. It also follows that if both v and $-v$ are in the fan then λ_v extends to a morphism $\mathbb{P}^1 \to X(\Delta)$ (with metric topology $\mathbb{P}^1 \cong \mathbb{S}^2$).

Examples

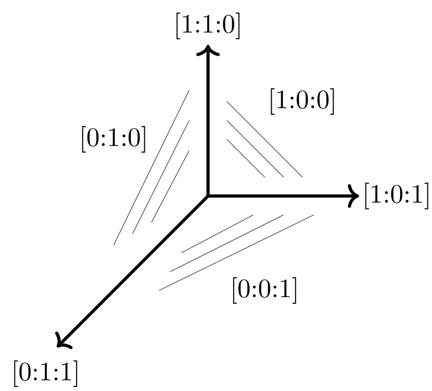
We now compute the fan associated to $\mathbb{P}^2$. We first embed $(\mathbb{C}^*)^2 \hookrightarrow \mathbb{P}^2$ via $(t_1, t_2) \mapsto [1 : t_1 : t_2]$. So we get

$$\lim_{z \to 0} \lambda_v(z) = \lim_{z \to 0} [1 : z^{v_1} : z^{v_2}]$$

We notice that if both $v_1 > 0$ and $v_2 > 0$ then the limit exists and has distinguished point $\lim_{z\to 0}[1 : z^{v_1} : z^{v_2}] = [1 : 0 : 0]$ so we know the first quadrant is part of one and the same cone that could potentially be bigger, but we don't know yet. If we set $v_1 < 0$ we notice that the limit of $[1 : v^{v_1} : v^{v_2}] = [v^{-v_1} : 1 : z^{v_2 - v_1}]$ exists if also $v_2 > v_1$, namely

$$\lim_{z \to 0} \lambda_v [z^{-v_1} : 1 : z^{v_2 - v_1}] = [0 : 1 : 0]$$

Changing v_1 and v_2 we see that the area with $v_2 < 0$ and $v_1 > v_2$ is part of a cone with distinguished point $[0 : 0 : 1]$. Since only three rays are left we know that these are exactly the interiors of all top dimension cones with the rays as facets. All that is potentially left is to calculate the distinguished points of these rays. For $v_1 = 0$ and $v_2 > 0$ we get that $[1 : 1 : z^{v_2}]$ goes to $[1 : 1 : 0]$ and similarly $[1 : 0 : 1]$ is the distinguished point for the $v_1 > 0$ and $v_2 = 0$ ray. Finally for $v_1 = v_2 < 0$ we get that $[1 : z^{v_1} : z^{v_1}] = [z^{-v_1} : 1 : 1] \to [0 : 1 : 1]$.



References

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