The Orbit-Cone Correspondance

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December 6, 2023

1 Introduction

In this talk we will discuss the so-called Riemann-Roch theorem for (potentially singular) toric varieties. To give some context, we will first discuss the classical Riemann-Roch theorem for line bundles over curves.

This theorem in its most basic form gives an expression for the dimension of the $\mathbb{C}$-vector space of meromorphic functions on a compact Riemann surface whose singularities are bounded by some fixed divisor. To see why the computation of this dimension is important, it already suffices to consider the trivial case $D = 0$ on the Riemann sphere S^2. Namely, the set of meromorphic functions which have poles no worse than D are then exactly the holomorphic functions $S^2 \to \mathbb{C}$. It is an important theorem from elementary complex analysis that any such function is constant. Let us look at some other examples on the Riemann sphere.

Example 1.1. *Set $D = \{0\}$ the divisor supported at the origin of $\mathbb{C} \subseteq S^2$. Then a meromorphic function f which has poles no worse than D is by definition a meromorphic function which has at most a single pole at the origin. Again, we obviously have the constant functions. But now, we also have z^{-1}, so the dimension of the space of such functions is $\ell(D) = 2$. What about if we add a second point $D = \{0, \infty\}$? Then we have the constant functions, z and z^{-1} and hence $\ell(D) = 3$. You might guess now that for a general divisor D $\ell(D) = \deg(D) + 1$ (at least when $\deg(D) \geq 0$) and indeed this is the most elementary formulation of the Riemann-Roch theorem.*

The required language quickly becomes more complicated though if we try to generalize this statement. For example, if C is a proper non-singular algebraic curve, the Riemann-Roch theorem states that

$$\ell(D) - \ell(K - D) = \deg(D) - g + 1.$$

Here g is the genus of the curve, ituitively counting the number of "holes" in the corresponding Riemann surface. Moreover, K is the canonical divisor on C, defined to be the divisor corresponding to the line bundle Ω_C, the sheaf of differentials. In the case of $\mathbb{P}^1$, K is a divisor of degree 2 and you can check that this makes everything work out the way we discussed above.

A fundamental insight leading to the generalization to the higher-dimensional case was that the term on the left-hand side can be interpreted as the Euler characteristic of the line bundle, i.e. the alternating sum over the dimensions of its cohomology groups (this can be seen by applying Serre duality to show that $\ell(K-D)$ is in fact equal to the dimension of $H^1(\mathcal{O}(D))$). This lead to the so-called Hirzebruch-Riemann-Roch formula:

$$\chi(X,E) = \int ch(E) \cap Td(X).$$

Here, X is a complete variety and E is a vector bundle on X. We will now unpack this definition and try to understand how its statement reduces to the classical Riemann-Roch theorem in the special case that X is a proper curve and E is a line bundle defined by a divisor D.

First, by definition, the left-hand side is simply the Euler characteristic of E which we already discussed above. The integral sign on the right-hand side means that one takes the degree of the term which follows. Hence it remains to define the two terms $ch(E)$ and $Td(X)$. We start with the chern character $ch(E)$, for which we will only give a definition when $E = L$ is a line bundle. The general definition is then obtained by applying the splitting principal and requiring the chern class to be multiplicative.

Definition 1.1. Let L be a line bundle on a variety X. We define the chern character of L to be the element of $A_*(X)_{\mathbb{Q}}$ defined by

$$ch(L) := \exp(c_1(L)).$$

We proceed similarly for the definition of the Todd class $Td(X)$ of X, whose definition is however somewhat more involved. We start by defining the Todd class of a line bundle (again, requiring multiplicativity extends this definition to a general vector bundle).

Definition 1.2. Let L be a line bundle on a variety X. We define the Todd class of L to be the element of $A_*(X)_{\mathbb{Q}}$ defined by

$$td(L) := c_1(L)/(1-\exp(-c_1(L))).$$

In the non-singular case, the class $Td(X)$ is then defined to be $td(T_X)$, where T_X is the tangent bundle of X. However, this cannot immediately be generalized to the singular case, since in this case T_X may no longer actually be a vector bundle. We can fix this by observing that if $f : X' \to X$ is a proper birational morphism of non-singular varieties with $f_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and $R^i f_*(\mathcal{O}_{X'}) = 0$ for all $i > 0$ (e.g., f is the blowup of a variety at a smooth subvariety), then $f_*(Td(X')) = Td(X)$. Using the resolution of singularities and requiring this property to hold even in the singular case allows us to extend the definition of $Td(X)$ to arbitrary varieties.

That was a lot of work just to define the objects in the statement of the Hirzebruch-Riemann-Roch theorem! Now let's see how the left-hand side reduces to $\deg(D) - g + 1$ in the case that $X = C$ is a non-singular proper curve. In this case, all Chow groups in dimensions higher than 1 vanish, so we get:

$$ch(L) \cap Td(X) = (1 + c_1(L))(1 + \frac{1}{2}c_1(T_X)).$$

But T_X is simply the dual of Ω_X and hence $c_1(T_X) = -c_1(\Omega_X)$. That being said, we get:

$$ch(L) \cap Td(X) = 1 + c_1(L) - \frac{1}{2}c_1(\Omega_X).$$

Now $c_1(L)$ is precisely the divisor D defining L and similarly, $c_1(\Omega_X)$ is the divisor defining the canonical sheaf, which can be shown to have degree $2g - 2$. Hence, taking the degree, we get:

$$\deg\left(1 + c_1(L) - \frac{1}{2}c_1(\Omega_X)\right) = \deg(D) - g + 1$$

as required.

In the higher dimensional case, the Hirzebruch-Riemann-Roch theorem for a line bundle L, says:

$$\chi(X, L) = \sum_{k=0}^{n} \frac{1}{k!} \deg(c_1(L)^k \cap Td_k(X)).$$

The upshot of working with toric varieties is now that, at least in the non-singular case, we have an explicit way of computing Ω_X, as we saw a few lectures ago. Namely, for X a non-singular toric variety, we have the relation:

$$c(\Omega_X^1) \cdot \prod_{i=1}^{d} c(\mathcal{O}_{D_i}) = 1,$$

coming from the short exact sequence:

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \to \bigoplus_{i=1}^{d} \mathcal{O}_{D_i} \to 0,$$

where D_i are the boundary divisors of X (Note that we are using the more general definition of a chern class of the coherent sheaves $\mathcal{O}_{D_i}$ here. This is well-defined only when X is non-singular). We can use this to show the following:

Lemma 1.1.

$$c(T_X) = \prod_{i=1}^{d}(1 + D_i) = \sum_{\sigma \in \Delta} [V(\sigma)].$$

Therefore, by definition of the Todd class:

$$td(T_X) = \prod_{i=1}^{d} \frac{D_i}{1 - \exp(-D_i)}.$$

Proof. First we start with the short exact sequence associated with the effective divisor D_i:

$$0 \to \mathcal{O}(-D_i) \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0.$$

From this and using the fact that $c(\mathcal{O}(-D_i)) = 1 - [D_i]$, we obtain:

$$c(\mathcal{O}_{D_i}) = (1 - [D_i])^{-1}.$$

Therefore:

$$c(\Omega^1_X) = \prod_{i=1}^{d}(1 - [D_i]).$$

Since $T_X = \Omega^\vee_X$, using the relationship $c_i(E^\vee) = (-1)^i c_i(E)$ shows:

$$c(T_X) = \prod_{i=1}^{d}(1 + [D_i]).$$

□

Note however, that the D_i need actually not be Chern roots of T_X, since there may be more than n of them. Using the relation $Td(X(\Delta)) = f_*(X(\Delta'))$ for a proper birational map $f : X(\Delta') \to X(\Delta)$, then gives us a method for computing the Todd class of an arbitrary toric variety (which is far from given for an arbitrary variety!).

Having developed all this algebraic language, we will now use it to show some non-trivial facts coming from combinatorics. Namely, as we saw when we discussed line bundles on toric varieties, for a T-Cartier divisor D which is generated by its sections, the higher cohomology groups vanish and the space of sections is the number of lattice points in a certain convex polytope $P = P_D$, whose vertices line in the lattice M. We denote this number by $\#(P) := |P \cap M|$. The Hirzebruch-Riemann-Roch theorem now tells us:

$$\#(P) = \sum_{k=0}^{n} \frac{1}{k!} \deg(D^k \cap Td_k(X)). \tag{*}$$

In fact, every convex rational polytope arrises as P_D for some T-Cartier divisor on some toric variety. Namely, we can always find a complete fan which contains the hyperplanes parallel to the faces of P so that we can find some divisor which corresponds to P. One can even relate the right-hand side to the volume of the Polytope, by observing that

$$Vol(P) = \lim_{\nu \to \infty} \frac{\#(\nu P)}{\nu^n}.$$

Namely, since the polytope $\nu P_D = P_{\nu D}$, this means that

$$\begin{aligned} Vol(P) &= \nu^n \lim_{\nu \to \infty} \sum_{k=0}^{n} \frac{\nu^k}{k!} \deg(D^k \cap Td_k(X)) \\ &= \lim_{\nu \to \infty} \frac{1}{n!} \deg(D^n \cap Td_n(X)) \\ &= \frac{\deg(D^n)}{n!}. \end{aligned}$$

From this, even without any deeper analysis, we get the following (mildly) non-obvious statement about the volume of convex polytopes:

Proposition 1.1. *Let $P \subseteq \mathbb{R}^n$ be a convex rational polytope with vertices in the integer lattice. Then $n! \cdot Vol(P)$ is an integer.*

In low dimensions, further inspection of (*) leads to more refined results. For instance:

Theorem 1.1 (Pick's formula). *Let $P \subseteq \mathbb{R}^2$ be a convex rational polygon. Then:*

$$\#(P) = Area(P) + \frac{1}{2} \cdot Perimeter(P) + 1$$

where the length of an edge of P is calculated in the quotient lattice of $\mathbb{Z}^2$.

For the proof we will need to have a more detailed description of the Todd class:

Lemma 1.2. *Let X be a toric variety with boundary divisors $D_1, \ldots, D_n$. Write $Td(X) = Td_n(X) + Td_{n-1}(X) + \cdots + Td_0(X)$, where $Td_k(X) \in A_k(X)_{\mathbb{Q}}$. Then the following statements hold:*

1. *$Td_n(X) = [X]$*
2. *$Td_{n-1}(X) = \frac{1}{2} \sum [D_i]$, where D_i are the boundary divisors of X.*
3. *If $dim(X) = 2$, then $Td_0(X) = [x]$*

Proof. We will deduce all of these facts starting from the definition of the Todd class in the non-singular case:

$$Td(X) = td(T_X) = 1 + \frac{1}{2} c_1(T_X) + \cdots + \frac{1}{n!} c_1(T_X)^n.$$

Here, $1 = [X]$ and thus the first statement is clear when X is non-singular. For the general case, let X' be a toric variety and pick a proper birational morphism $\phi : X \to X'$ from some non-singular toric variety X, such that $\phi_*(\mathcal{O}_X) = \mathcal{O}_{X'}$ (this exists by desingularization). The second fact is precisely the statement that $\phi_*([X]) = [X']$, whence the first claim.

For the second claim, we must show that $c_1(T_X) = \sum[D_i]$ when X is a non-singular variety. This in fact follows immediately from Lemma 1.1, by expanding the right-hand side. For a general X', pick $\phi_* : X \to X'$ as above. But we have seen, that for any proper morphism of toric varieties, $f_*[V(\sigma)] = [V(\sigma')]$, if σ is contained in some cone σ of equal dimension and $f_*[V(\sigma)] = 0$ otherwise. Since X is a subdivision of X', it follows at once, that $f_*(\sum[D_i] = \sum[D_i']$.

Finally, for the last statement, in the non-singular case we can simply compute the entire expansion of $Td(X)$ (which has only terms up to degree 2). This yields:

$$\begin{aligned}
Td(X) &= \prod_{i=1}^{d} \frac{D_i}{1-\exp(-D_i)} \\
&= \prod_{i=1}^{d} \frac{D_i}{1-1+D_i-\frac{1}{2}D_i^2+\frac{1}{6}D_i^3} \\
&= \prod_{i=1}^{d} \frac{1}{1-\frac{1}{2}D_i+\frac{1}{6}D_i^2} \\
&= \prod_{i=1}^{d} \sum_{n=0}^{\infty} (\frac{1}{2}D_i - \frac{1}{6}D_i^2)^n \\
&= 1 + \frac{1}{2}\left(\sum_{i=1}^{d} D_i\right) + \frac{1}{4}\left(\sum_{\{i,j\}} D_i \cdot D_j\right) - \frac{1}{6}\left(\sum_{i=1}^{d} D_i^2\right) \\
&= 1 + \frac{1}{2}\left(\sum_{i=1}^{d} D_i\right) + \frac{1}{4}\left(\sum_{\{i,j\}, i\neq j} D_i \cdot D_j\right) + \frac{1}{12}\left(\sum_{i=1}^{d} D_i^2\right).
\end{aligned}$$

Now in fact, it can be checked that $D_i^2 = -a_i[x]$, where $a_i v_i = v_{i-1} + v_{i+1}$, using the results from last week. Moreover, these a_i satisfy the equation

$$\sum_{i=1}^{d} a_i = 3d - 12.$$

Finally, $\sum_{i<j} D_i \cdot D_j = d[x]$, since each cone appears exactly once. Hence:

$$Td(X) = 1 + \frac{1}{2}\left(\sum_{i=1}^{d} D_i\right) + [x].$$

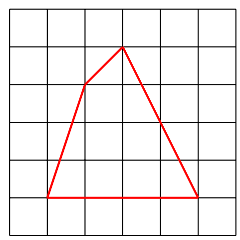
The result for general X follows from the fact that if $X \to X'$ is birational, then the preimage of a generic point is a single point. □

Proof. Let $X(\Delta)$ be a non-singular variety with a divisor D corresponding to P. By Lemma 1.2, we have $Td(X) = [X] + \frac{1}{2}\sum_{i=1}^{d}[D_i] + [x]$ for some $x \in X$. Thus dimension 2, (*) reads:

$$\#(P) = \deg([x]) + \deg\left(\frac{1}{2}\sum_{i=1}^{d} D \cap D_i\right) + \frac{1}{2}\deg(D^2 \cap Td_2(X)).$$

By the expression found above, the last term is precisely the area of P. Moreover, the first term is 1. For the second term, observe that $D \cap D_i$ corresponds to an edge of P (or 0 if D and D_i do not span a cone) and $\deg(D \cap D_i)$ is thus the length of this edge in the quotient lattice. □

Example 1.2. *Consider the polygon given below:*



We have:

$$\begin{aligned} \mathit{Area}(P) &= \frac{19}{2} \\ \mathit{Perimeter}(P) &= 7 \\ \#(P) &= 14 \end{aligned}$$

and indeed:

$$\frac{19}{2} + \frac{7}{2} + 1 = 14.$$