Talk 2:

- If $\mathcal{L}$ is a line bundle on a variety X and σ is a rational section, then on an open affine U, we may write σ of the form $\frac{f_U}{g_U}$ and we define $\mathrm{Div}(\sigma)|_U = \mathrm{Div}(f) - \mathrm{Div}(g)$

$\left(\mathrm{Div}\, f = \sum_{Y \subseteq X} \mathrm{ord}_Y(f) \langle Y \rangle\right)$

Furthermore, if τ is another section of $\mathcal{L}$, then $\alpha = \sigma/\tau$, so $\mathrm{Div}(\sigma) - \mathrm{Div}(\tau) = \mathrm{Div}(\alpha) \equiv 0$ (mod) $\mathrm{Rat}\, X$

We define the first Chern class on a line bundle to be $c_1(\mathcal{L}) \in A(X)$ to be the rational equivalence of the divisor σ for any non-zero σ.

Example : $c_1(\mathcal{O}_{\mathbb{P}^n}(d))$ is the hyper surface of degree d.

proof that $c_1(\mathcal{L}) = [\mathrm{Div}(\sigma)]$, the rational equivalence of any divisor σ.

Recall that a cycle is rationally equivalent to 0 iff, if there is $\{W_i\}_{i \in [n]}$ subschemes of X and $r_i \in R(W_i)^*$ such that $\alpha = \sum [\mathrm{div}(r_i)]$.

If σ, τ are two rational section then $\alpha = \sigma / \tau$ is well defined and $\mathrm{Div}(\alpha)$ is rationally equivalen to 0 ~~same~~ by definition

Chern Classes of Vector Bundles

To define the Chern class of a vector bundle we are motivated by the following fact:

There is a unique way of assigning to each vector bundle ξ on a smooth quasi projective variety X a class $c(\xi) = 1 + c_1 + \cdots \in A(X)$ in such a way that:

(a) (Line bundles) If $\mathcal{L}$ is a line bundle on X then the chern class of $\mathcal{L}$ is $1 + c_1(\mathcal{L})$ where $c_1(\mathcal{L}) \in A^1(X)$

(b) (Whitney's formula) If

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$ is a

short exact sequence of vector bundles

then $c(\mathcal{F}) = c(\mathcal{E})\, c(\mathcal{G}) \in A(X)$

(c) Functoriality, if $\varphi : Y \longrightarrow X$ is a

morphism of smooth varieties then

$$\varphi^*(c(\mathcal{E})) = c(\varphi^*(\mathcal{E}))$$

(d) $c_i(E^\vee) = (-1)^i\, c(E)$

If $f: X \longrightarrow Y$ is a proper morphism, then

for $V \subseteq X$ subvariety, $f(V)$ is a subvariety of Y

we then have that $R(f(V)) \hookrightarrow R(V)$ and

so $f_*[V] = [R(V) : R(f(V))][f(V)]$

Pullback

If $f: Y \longrightarrow X$ is a ~~flat~~ morphism, then there is

a unique map of groups $f^*: A^c(X) \longrightarrow A^c(Y)$

so that whenever $A \subseteq X$ is a subvariety

generically transverse to f, we have that.

$$f^*([A]) = [f^{-1}(A)]$$

The Splitting Construction

If X is a smooth variety and $\mathcal{E}$ a vector bundle of rank r on X, then there exists a smooth variety Y and a morphism

$$\varphi : Y \longrightarrow X$$ (Y is the flag variety)

with the following two properties.

(a) The pullback map $\varphi^* : A(X) \longrightarrow A(Y)$ is injective

(b) The pull back bundle $\varphi^*\mathcal{E}$ on Y admits a filtration

$$0 = \xi_0 \subseteq \xi_1 \subseteq \cdots \subseteq \xi_{r-1} \subseteq \xi_r = \varphi^*\xi$$

of vector subbundles with successive quotients ξ_i/ξ_{i-1} locally free of rank 1.

Application: If ξ is a vector bundle of rank r, then $c_i(\xi) = 0$ for $i > r$. The reason being that if ξ split as $\bigoplus_{i=1}^{r} \mathcal{L}_i$ for line bundles $\mathcal{L}_i$, then since $c(\mathcal{L}_i) = 1 + c_1(\mathcal{L}_i)$

$$c(\xi) = \prod_{i=1}^{r} (1 + c_1(\mathcal{L}_i))$$

Example: If $\xi = \bigoplus \mathcal{L}_i$ then

$$c(\xi^\vee) = \prod 1 + c_1(\mathcal{L}_i^*) = \prod 1 - c_1(\mathcal{L}_i)$$

since $c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L}^\vee)$ because the dual line bundle is isomorphic to the inverse line bundle

Euler Sequence, and $c(T_{\mathbb{P}^n})$

The euler sequence is the following exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

Now, by the whitney theorem we have that

$$c(\mathcal{O}(1)^{n+1}) = c(\mathcal{O}) \times c(T_{\mathbb{P}^n})$$

$$c(\mathcal{O}(1)^{n+1}) = c(T_{\mathbb{P}^n})$$

$$= (1 + H)^{n+1}$$

where $H \in A^1(\mathbb{P}^n)$ is the hyper plane class

Recall that the sheaf $\mathcal{O}(1)$ is generated by the global sections, and the sections correspond to hyperplane, because the global sections of $\mathcal{O}(1)$ are degree 1 homogeneous polynomials.

$\mathcal{O}(1)(X) = \langle x_0, \dots, x_n \rangle$, and recall that any two hyperplanes represent the same non-trivial class in $Cl(X)$.

If H_1 is the zero set of $a_0x_0 + \dots + a_nx_n$ and H_2 is the zero set of $b_0x_0 + \dots + b_nx_n$, then $H_2 - H_1 = (a_0x_0 + \dots + a_nx_n)/(b_0x_0 + \dots + b_nx_n)$ and this is zero.

Claim: $c(T_X) = \sum_{\sigma} [V(\sigma)]$

pf:

Recall that $c(T_X) \in A_0(X) \oplus \cdots \oplus A_n(X)$

The chern classes of the tangent bundle T_X of a smooth complete toric variety are called the chern classes of X

definition: The top chern class is $c_n(T_X)$ where n is the dimension of X.

The cotangent bundle of a smooth complete toric variety fits into the

Euler sequence: (Recall that $T_X = \Omega^{\vee}$)

$$0 \longrightarrow \Omega^1_X \longrightarrow \bigoplus_\rho \mathcal{O}_X(-D_\rho) \longrightarrow \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \to 0$$

The total chern class of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X$ is 1

$$\Rightarrow c(\Omega^1_X) = c\Big(\bigoplus_\rho \mathcal{O}_X(-D_\rho)\Big)$$

$$= \prod_\rho c(\mathcal{O}_X(-D_\rho)) = \prod (1-[D_\rho]$$

(Chern classes of the tangent bundle T_X of a smooth toric variety are called the chern classes of X)

def^n: The top chern class is $c_n = c_n(T_X)$

where $n = \dim X$ and this is usually called the Euler class.

Prop:

(a) $c(T_X) = \prod_\rho (1+[D_\rho])$

Consider the dual of the exact sequence from the previous proof.

We have that:

$$\begin{tikzcd} & & & \mathcal{O}_X^{r-n} \arrow[d, "\cong"] & \\ 0 \arrow[r] & T_X \arrow[r] & \left(\bigoplus_\rho \mathcal{O}_X(-D_\rho)\right)^\vee \arrow[r] & \left(\mathrm{Pic}(X)\otimes \mathcal{O}_X\right)^\vee \arrow[r] & 0 \end{tikzcd}$$

Now by spamming the fact that ξ of

$$c_i(\xi^\vee) = (-1)^i c_i(\xi)$$

we get:

$$c(T_X) = \prod_\rho (1 + [D_\rho]).$$

Now $\prod_\rho (1 + D_\rho) = \sum_{\sigma \in \Sigma} [V(\sigma)]$ by the orbit cone correspondence which

<u>Lemma 12.5.2</u>:

$$[D_{\rho_1}] \cdot \ldots \cdot [D_{\rho_d}] = \begin{cases} \frac{1}{\text{mult } \sigma} [V(\sigma)] \\ 0 \text{ otherwise} \end{cases}$$

$$\prod_\rho (1 + D_\rho) = 1 + 0 + \cdots D$$

Degree map & the Integral

If X is a complete ~~map~~ scheme, i.e. a scheme that is proper over $\operatorname{Spec}(\mathbb{C})$ and $\alpha = \sum_p n_p [p]$ is a zero cycle on X, then the degree of α, which we denote by $\deg(\alpha) = \int_X \alpha = \sum_p n_p [R(p) : K]$

degree of field extn.

where $R(p)$ is defined to be field of rational functions.

The degree map satisfies the following properties:

(a) Rationally equivalent cycles have the same degree

(c) The degree map is equivalent to the statement that $\deg(\alpha) = p_* \alpha$, where $p : X \longrightarrow \mathrm{Spec}\, k$ is the structural morphism, and we identify $A_* S = \mathbb{Z}[S]$

Remark: we can extend the degree homomorphism to all of $A_* X$ by insisting that:

$$\int_X \alpha = 0 \text{ if } \alpha \in A_k X, \ k > 0.$$

- The degree map satisfies the following useful property: for any morphism $f : X \longrightarrow Y$ and $\alpha \in A_* X$ we have that $\int_X \alpha = \int_Y f_* \alpha.$

$$\int_X c_n(T_X) = e(X)$$

where $e(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H^i(X)$