Our goal is to find out the structure of the CHOW-Ring.

- As we have seen: if $\sigma = \mathrm{cone}(v_1, \dots, v_k)$ then

  $V(\sigma)$ is the transversal intersection of $D_1, \dots, D_k$.

$\Rightarrow A^* X$ generated as $\mathbb{Z}$-algebra by $\{D_\rho \mid \rho \in \Delta(1)\}$
& is a graded ring via intersection product.

- We have relations:

intersection product

-> $D_1 \cdots D_k = 0$ in $A^k X$ if $\mathrm{cone}(v_1, \dots, v_k) \notin \Delta$ (*)

-> if $u \in M$ then $\mathrm{div}(\chi^u) = \sum \langle u, v_i \rangle D_i = 0_{A^1 X}$.

We can prove a converse:

<u>Proposition</u> For a smooth toric (*) ordered variety X_Δ.

$$A^* X = \mathbb{Z}[D_1, \dots, D_d] / I$$

where I is generated by

$$\begin{cases} D_{i_1} \cdots D_{i_k} & \text{for } \mathrm{cone}(v_{i_1}, \dots, v_{i_k}) \notin \Delta \\ \text{and} \quad \sum_{i=1}^d \langle u, v_i \rangle D_i & \text{for } u \in M \end{cases}$$

where $D_1, \dots, D_d$ are the divisors corresponding to $\Delta(1) = \{\rho_1, \dots, \rho_d\}$, $\rho_j = \mathrm{cone}(v_j)$.

- <u>Comments</u>:

-> it is enough to take $u \in M$ in a basis for M.

-> $\sum_{i=1}^d \langle u, v_i \rangle D_i = \mathrm{div}(\chi^u)$ because $\langle u, v_i \rangle \overset{\text{Thm.}}{=} \nu_{D_i}(\chi^u)$

-> $V(\sigma) \cap V(\tau) = \begin{cases} V(\gamma) & \text{if } \mathrm{cone}(\sigma \cup \tau) = \gamma \in \Delta \\ \underset{0 \text{ as cycle}}{\emptyset} & \text{if } \mathrm{cone}(\sigma \cup \tau) \notin \Delta \end{cases}$ as varieties.

-> $\mathbb{Z}[D_1, \dots, D_d] / \mathbb{Z} \cdot \{D_{i_1} \cdots D_{i_k} \mid \mathrm{cone}(v_{i_1}, \dots, v_{i_k}) \notin \Delta\} \xrightarrow{\sim} PP(\Delta)$

so the thm reads $A^* X \simeq PP(\Delta)/M$

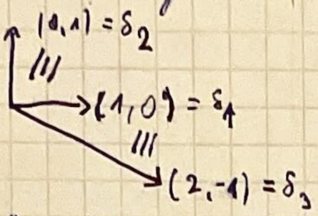
in particular $A^1 X \simeq PL(\Delta)/M$.

(*) $V(\sigma) \cap V(\tau)$ is proper $\Leftrightarrow \dim \overset{\mathrm{cone}(\sigma \cup \tau)}{\overset{\shortparallel}{\gamma}} = \dim \tau + \dim \sigma$

in this case $V(\sigma)$ meets $V(\tau)$ transversally in $V(\gamma) \Rightarrow V(\sigma) \cdot V(\tau) = \begin{cases} V(\gamma), & \in \Delta \\ 0, & \notin \Delta \end{cases}$

Example I

Consider the fan Δ that is a resolution of singularities of $Z(xy - z^2)$.



	$\langle e_1^*, \cdot \rangle$	$\langle e_2^*, \cdot \rangle$
$\delta_1 = e_1$	1	0
$\delta_2 = e_2$	0	1
$\delta_3 = 2e_1 - e_2$	2	-1

We obtain the relations

$$\text{generate } I. \left\{ \begin{array}{l} \delta_1 \delta_2 \delta_3 \in I \\ \delta_1 + 2\delta_3 \in I \\ \delta_2 - \delta_3 \in I \\ \delta_2 \delta_3 \in I \end{array} \right.$$

$A^0(X) = \mathbb{Z} \cdot [X]$

$A^1(X)$ has generators $\cancel{\delta_1}, \delta_2, \cancel{\delta_3} = \delta_2$

$$-2\delta_2 = \delta_1$$

$$\Rightarrow A^1(X) = \mathbb{Z} \cdot [\delta_2].$$

$A^2(X)$ has generators $\cancel{\delta_1^2}, \delta_2^2, \underbrace{\cancel{\delta_3^2}}_{=\delta_2^2}, \underbrace{\cancel{\delta_1 \delta_2}}_{=-2\delta_2^2}, \underbrace{\cancel{\delta_2 \delta_3}}_{=0}, \underbrace{\cancel{\delta_1 \delta_3}}_{=\delta_1 \delta_2}$

$$\delta_1^2 = (-2\delta_2)^2 = 4\delta_2^2$$

$$A^2(X) = \mathbb{Z} \cdot [\underbrace{\delta_2^2}_{= \delta_2 \delta_3 = 0}] = 0$$

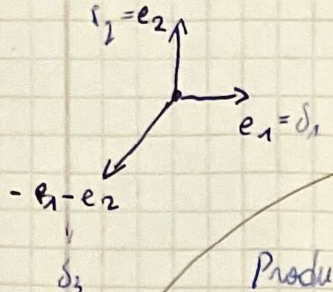
$A^3(X)$ has generators $\delta_i \delta_j \delta_k = \delta_1^r \delta_2^s, \; r + s = 3$

$$= (-2)^r \delta_2^{r+s}$$

$$= (-2)^r \delta_2^3 = 0$$

$$\Rightarrow A^* X = \mathbb{Z} \oplus \mathbb{Z}[\delta_2]$$

Example I: We consider the smooth toric variety $X_\Delta \xrightarrow{\sim} \mathbb{P}^2$ where Δ is:



$A^* \mathbb{P}^2$ has generators $\delta_1, \delta_2, \delta_3$ as a $\mathbb{Z}$-algebra, that satisfy some relations.

divisor corresponding to $\rho_1 = \mathrm{cone}(e_1)$

Products: $\delta_1 \delta_2 \delta_3 \in I$

Differences: (e_1^*, e_2^*) basis for M.

	$\langle e_1^*, \cdot \rangle$	$\langle e_2^*, \cdot \rangle$
e_1	1	0
e_2	0	1
$-e_1-e_2$	-1	-1

Hence the relations:
$$\delta_1 - \delta_3 \in I$$
$$\delta_2 - \delta_3 \in I$$

$$A^0(X) = \mathbb{Z} \cdot [\mathbb{P}^2] \cong \mathbb{Z}$$

$$A^1(X) = \{ a_1 \delta_1 + a_2 \delta_2 + a_3 \delta_3 \mid a_i \in \mathbb{Z} \}$$
$$= \mathbb{Z} \cdot \delta_1$$

$$A^2(X) = \{ \sum_{i+j+k=2} a_{ijk} \delta_1^i \delta_2^j \delta_3^k \mid a_{ijk} \in \mathbb{Z} \}$$

has generators as groups: $\delta_1^2, \cancel{\delta_2^2}, \cancel{\delta_3^2}, \delta_1\delta_2, \delta_1\delta_3, \delta_2\delta_3$
$$\hat{=} \quad \delta_1^2,$$

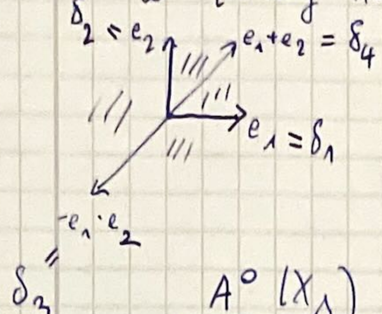
$$\Rightarrow A^2(X) = \mathbb{Z}[\delta_1^2]$$

$$A^3(X) = \{ \sum_{i+j+k=3} a_{ijk} \delta_1^i \delta_2^j \delta_3^k \mid a_{ijk} \in \mathbb{Z} \}$$
$$= \{0\} \quad \text{because } \delta_1^i \delta_2^j \delta_3^k = \delta_1^{i+j+k} = \delta_1^3 = \delta_1 \delta_2 \delta_3 = 0$$

$$A^4(X) = \{0\} = A^5(X) = \dots$$

Hence $A^*(X) = \mathbb{Z}[\delta_1^2] \oplus \mathbb{Z}[\delta_1] \oplus \mathbb{Z}[\mathbb{P}^2]$ as $\mathbb{Z}$-algebra.

Example II We consider the star-refinement of $\mathbb{P}^2$ at the ray through e_1+e_2; the fan looks like this:



so we have a morphism $(X_\Delta, \mathcal{O}_{X_\Delta}) \to (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$

$A^*(X_\Delta)$ has generators $\delta_1, \dots, \delta_4$ as $\mathbb{Z}$-alg

$\underline{A^0(X_\Delta) = \mathbb{Z}\cdot[X_\Delta]}$

$A^1(X_\Delta)$ has generators $\delta_1, \delta_2, \delta_3, \delta_4$ as a group.

	$\langle e_1^*, \cdot\rangle$	$\langle e_2^*, \cdot\rangle$
e_1	1	0
e_2	0	1
e_1+e_2	1	1
$-e_1-e_2$	-1	-1

$\rightsquigarrow$ relations $\delta_1+\delta_3-\delta_4 = 0$, $\delta_2+\delta_3-\delta_4 = 0$

$\underline{\delta_3\delta_4 = 0}$, $\underline{\delta_1\delta_2 = 0}$, $\underline{\delta_i\delta_j\delta_k = 0 \ \forall i,j,k}$.

$0 = \delta_2+\delta_3-(\delta_3+\delta_1) = \delta_2-\delta_1$

$\Rightarrow \delta_1 = \delta_2$, $\delta_1+\delta_3-\delta_4 = 0 = \delta_1+\delta_3-\delta_4$

$A^1(X_\Delta) = \mathbb{Z}\cdot[\delta_1]\oplus\mathbb{Z}[\delta_3]\oplus\mathbb{Z}[\delta_4] \Big/ \delta_1 = \delta_4-\delta_3$

$\underline{= \mathbb{Z}[\delta_1]\oplus\mathbb{Z}[\delta_3]}$

$\delta_1^2 = 0$, $\delta_3\delta_4 = 0 = \delta_3(\delta_1+\delta_3) = \delta_3\delta_1+\delta_3^2$

$\delta_1(\delta_4-\delta_3) = 0 = \delta_1\delta_4-\delta_1\delta_3 = \delta_1\delta_4-\delta_3^2$

$$\begin{array}{cccccc} \cancel{\delta_1^2} & \delta_3^2 & \delta_4^2 & \delta_1\delta_3 & \delta_1\delta_4 & \cancel{\delta_3\delta_4} \\ 0 & \| & \| & \| & \| & 0 \\ & -\delta_1\delta_3 & \delta_3^2 & -\delta_3^2 & \delta_3^2 & \end{array}$$

$\delta_4^2 = \delta_4(\delta_1+\delta_3) = \delta_1\delta_4+\delta_3\delta_4 = \delta_3^2+0$

$\underline{\Rightarrow A^2(X_\Delta) = \mathbb{Z}\cdot[\delta_3^2]}$

$\underline{A^3(X_\Delta) = 0}$

Hence $A^*(X_\Delta) = \mathbb{Z}[X_\Delta]\oplus\mathbb{Z}[\delta_1]\oplus\mathbb{Z}[\delta_3]\oplus\mathbb{Z}[\delta_3^2]$.

Consider $T = \delta_1^p\delta_3^q$ w/ $p+q=3$ then
$p=3 \Rightarrow T = \delta_1^3 = 0$
$p=2 \Rightarrow T = 0 = \delta_1^2$
$p=1 \Rightarrow T = \delta_1\delta_3^2 = +\delta_1^2\delta_4 = 0$
$p=0 \Rightarrow T = \delta_3\delta_3^2 = -\delta_3\cdot\delta_3\delta_1 = -\delta_3^2\delta_1 = 0$

proof of the Proposition $\quad X := X_\Delta$ of $\dim X = n$.

Let $A^\bullet = \mathbb{Z}[D_1, \dots, D_d] / I$ with D_j regarded as X_j in a polynomial ring. We have

$$q : \mathbb{Z}[D_1, \dots, D_d] \longrightarrow \bigoplus_p A^p(X) = \bigoplus_p \langle V(\sigma) \mid \operatorname{codim}\sigma = p \text{ in } \Delta \rangle / \text{rational}$$

$$D_i \longmapsto [D_i]$$

$$(1 \longmapsto X)$$

is surjective using some of the previous observations.

Furthermore $I \subset \ker q$ because

we saw that $D_{i_1} \cdots D_{i_k} = 0_{A^k X}$ if $\operatorname{cone}(v_{i_1}, \dots, v_{i_k}) \notin \Delta$

and sums $\sum \langle u, v_i \rangle D_i$ for $u \in M$ correspond to $\underbrace{\operatorname{div}(\chi^u)}_{\text{vanish in } A^1 X} \in \ker q$ by assumption.

$\Rightarrow$ surjective map $A^\bullet \longrightarrow\!\!\!\!\rightarrow \bigoplus_p A^p(X)$

If $\sigma = \operatorname{cone}(v_{i_1}, \dots, v_{i_k})$ set $p(\sigma) = D_{i_1} \cdots D_{i_k}$

Lemma: There is an enumeration of the maximal cones of Δ $\sigma_1, \dots, \sigma_m$ and a finite sequence $\tau_1, \dots, \tau_m$ w/

$$\begin{cases} \tau_i \subset \sigma_i & \forall i \\ \tau_i = \bigcap\limits_{\substack{j > i \\ \dim(\sigma_i \cap \sigma_j) = n-1}} \sigma_j & \\ \tau_i \not\subset \sigma_j & \forall i < j \end{cases}$$

Last Step: $A^\bullet \longrightarrow \bigoplus_p A^p$ is injective. Enough: $p(\tau_1), \dots, p(\tau_m)$ generate $A^\bullet$ as a $\mathbb{Z}$-module.

Even if this is not completely obvious, we will generalize the trick that we used in the examples:

- we move the repetitions away as in the examples

$$\delta_3^2 \delta_1 \rightsquigarrow \delta_2 \delta_4 \delta_1 + \delta_5 \delta_2 \delta_1 \cdots$$

Algebraic Lemma

Let $\alpha \subsetneq \gamma < \beta$ cones in Δ, $k = \dim(\gamma)$.
Then there is an equation $p(\gamma) = \sum m_i \, p(\gamma_i)$ in $A^\bullet$
with γ_i cones of dimension k in Δ with $\alpha < \gamma_i$
but $\gamma_i \nless \beta$ and m_i integers

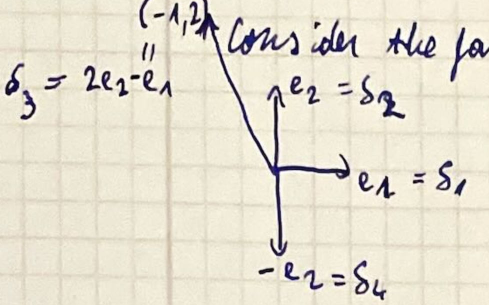
$\hookrightarrow$ By induction on the number of ray generators we can show that $A^\bullet$ is additively generated by monomials in $D_1, \dots, D_d$. $\quad A^\bullet = \mathbb{Z} \cdot \{ D_{i_1} \cdots D_{i_k} \mid k \geq 0 \}$

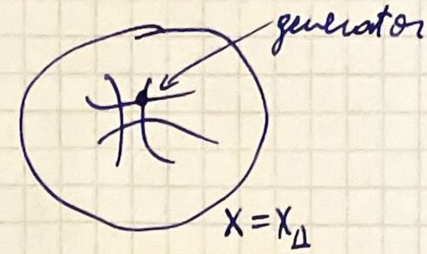
$\rightarrow$ By descending induction on i we show that if γ lies between τ_i and σ_i then
$p(\gamma) \in \mathbb{Z} \cdot \{ p(\tau_j) \mid j \geq i \}$

- if $\gamma = \tau_i$ then the claim follows
- if not, then lemma $\Rightarrow$
  $p(\gamma) = \sum m_t \, p(\gamma_t)$. Then by the inductive hypothesis and a little argument the proposition follows.

Example III

Consider the fan Δ:





A^*X has generators $\delta_1, \dots, \delta_4$ with relations.

$\delta_1\delta_3 \in I$, $\delta_2\delta_4 \in I$, $\delta_i\delta_j\delta_k \in I$ for pairwise different i,j,k

$\delta_1\delta_2\delta_3\delta_4 \in I$

	$\langle e_1^*, \cdot\rangle$	$\langle e_2^*, \cdot\rangle$
e_1	1	0
e_2	0	1
$-e_1+2e_2$	-1	2
$-e_2$	0	-1

$\delta_1 - \delta_3 \in I$.

$\delta_2 + 2\delta_3 - \delta_4 \in I$

$A^0(X) \cong \mathbb{Z}\cdot[X]$

$A^1(X)$ has generators $\delta_1, \underbrace{\cancel{\delta_2}}_{=\delta_4 - 2\delta_1}, \overset{=\delta_1}{\cancel{\delta_3}}, \delta_4$ as group.

$\Rightarrow A^1(X) \cong \mathbb{Z}\cdot[\delta_1] \oplus \mathbb{Z}\cdot[\delta_4]$

$A^2(X)$ has generators $\overset{\delta_1\delta_3 \in I}{\cancel{\delta_1^2}}, \cancel{\delta_2^2}, \cancel{\delta_4^2}, \delta_1\cancel{\delta_2}, \delta_1\delta_4, \cancel{\delta_2}\delta_4$

$\delta_2^2 = \delta_2\delta_4 - 2\delta_1\delta_2$

$\delta_1\delta_2 = \delta_1(\delta_4 - 2\delta_1) = \delta_1\delta_4 - 2\delta_1^2$

$0 = \delta_2\delta_4 = \delta_4(\delta_4 - 2\delta_1) = \delta_4^2 - 2\delta_1\delta_4 \Rightarrow \delta_4^2 = 2\delta_1\delta_4$

$\Rightarrow A^2(X) = \mathbb{Z}\cdot[\delta_1\delta_4]$

$A^3(X) = \{0\} = A^l(X) \quad \forall\, l \geq 3$.

So $A^*X = \mathbb{Z}\cdot[X] \oplus \mathbb{Z}[\delta_1] \oplus \mathbb{Z}[\delta_4] \oplus \mathbb{Z}[\delta_1\delta_4]$

if $\delta_i^2\delta_j \in A^3(X)$ then $\delta_i^2 = n\cdot\delta_1\delta_4 \Rightarrow \delta_i^2\delta_j = n\cdot\delta_1\delta_4\cdot(a\delta_1 + b\delta_4) = an\,\overset{0}{\delta_1^2\delta_4} + bn\,\delta_1\delta_4^2$

but $\delta_1\delta_4^2 = \delta_1(-2\delta_1\delta_4) = -2\underbrace{\delta_1^2}_{=\delta_1\delta_3=0}\delta_4 = 0$