Cohomology of the line bundle

401-3140-73L Toric Geometry

14.11.2023 HS23

We saw that a T-Cartier Divisor $D = \sum a_i D_i$ determines a piecewise linear function $\psi_D \in \mathrm{PL}(\Delta)$ on $|\Delta|$ by $\psi_D(v_i) = -a_i$, where v_i is the first lattice point of a ray in Δ. Conversely we get $[D] = \sum -\psi_D(v_i) D_i$. This now defines a rational convex polyhedron in $M_\mathbb{R}$ by

$$\begin{aligned} P_D &= \{u \in M_\mathbb{R} \mid \langle u, v_i \rangle \leq -a_i \, \forall i\} \\ &= \{u \in M_\mathbb{R} \mid u \leq \psi_D \text{ on } |\Delta|\} \end{aligned}$$

We also saw that the torus action gives a grading of the homology groups. For example for H^0, the global sections, we get

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^0(X, \mathcal{O}(X))_u$$

with

$$H^0(X, \mathcal{O}(D))_u = \begin{cases} \mathbb{C}\chi^u & \text{if } u \in P_D \cap M \\ 0 & \text{else} \end{cases}$$

To generalize this we first define for any $u \in M$ the following conical set

$$Z(u) = \{v \in |\Delta| \mid \langle u, v \rangle \geq \psi_D(v)\}$$

First note that $u \in P_D$ if and only if $Z(u) = |\Delta|$. With this we can now define the local cohomology groups

$$H^p_{Z(u)}(|\Delta|) = H^p(|\Delta|, |\Delta| \setminus Z(u); \mathbb{C})$$

which are just the singular cohomology groups of the pair $(|\Delta|, |\Delta| \setminus Z(u))$. Looking at the associated long exact sequence with reduced cohomology groups we get for $p > 0$

$$\cdots \to H^p_{Z(U)}(|\Delta|, |\Delta| \setminus Z(u)) \to \underbrace{\tilde{H}^p(|\Delta|)}_{0} \to \tilde{H}^p(|\Delta| \setminus Z(u)) \xrightarrow{\cong} H^{p+1}_{Z(U)}(|\Delta|, |\Delta| \setminus Z(u)) \to \underbrace{\tilde{H}^{p+1}(|\Delta|)}_{0} \to \cdots$$

For $p = 0$ we get nonzero local cohomology if $|\Delta| \setminus Z(u) = \emptyset$.

Proposition 1. *We have for $p \geq 0$ that*

$$H^p(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}(D))_u \quad \textit{with} \quad H^p(X, \mathcal{O}(D))_u = H^p_{Z(u)}(|\Delta|)$$

The idea of the proof is to use the Čech complex $C^\bullet$ using the affine cover $X(\Delta) = \bigcup X(\sigma)$ defined by

$$C^p = \bigoplus_{\sigma_0, \ldots, \sigma_p} \underbrace{\mathcal{O}(D)(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p})}_{H^0(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}, \mathcal{O}(D))}$$

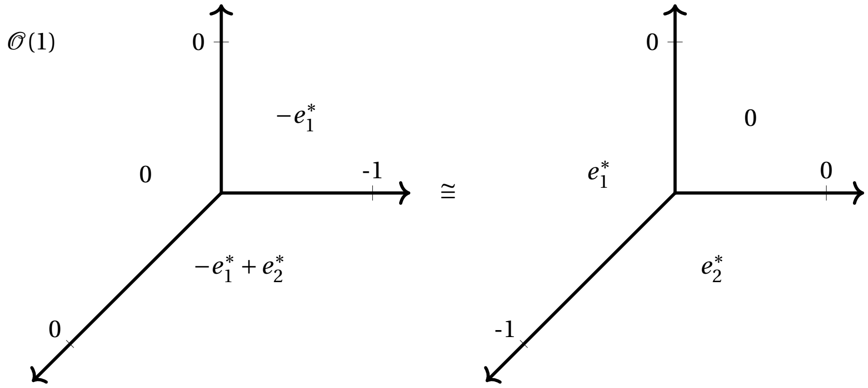
and then using a spectral argument.

Examples in $\mathbb{P}^1$

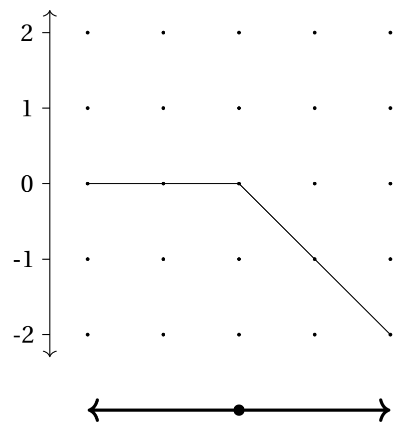
Before starting first a few words about $\mathbb{P}^n$. The fan of $\mathbb{P}^n$ can be realised as all basis vectors and negative their sum $\{e_1, \ldots, e_n, -\sum_i e_i\}$. Because this gives us $n+1$ rays where any selection of n primitive elements span all of $\mathbb{R}^n$, any piecewise linear function on $|\Delta|$ that is defined by its value on primitive ray elements can be made a linear function by changing its value on at most one ray giving us

$$\mathrm{Pic}(\mathbb{P}^n) \cong \mathrm{PL}(\Delta) / \{\text{linear functions}\} \cong \mathbb{Z}$$

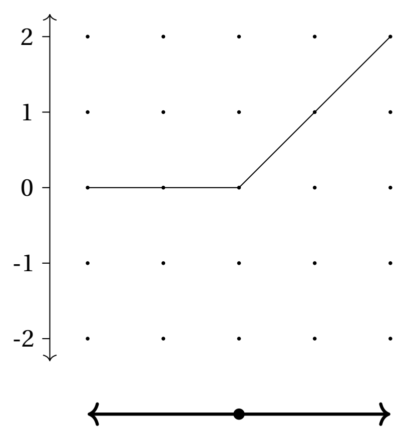
We can therefore represent a line bundle $\mathcal{O}(D)$, up to isomorphism, with an integer as $\mathcal{O}(n)$. Let $H \cong \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ a, in our case T-invariant, subvariety, then these line bundles correspond to a divisor nH and $\mathcal{O}(n) := \mathcal{O}(nH)$ to functions to $\mathbb{A}^1$ which have a pole of order at most n on H. On the fan they can be represented by a piecewise linear function that is $-n$ on one primitive ray element and 0 on the others.



If we now look at $\mathbb{P}^1$ the graph of a $e_i^* \in M$ is just a line going throw the origin and the point $(1, i)$. For $\mathcal{O}(1)$ we see that this line is above or on the graph of the representing function for $i \in \{-1, 0\}$ so $P_D = \{e^*_{-1}, 0\}$ meaning $H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C}\chi^{e^*_{-1}} \oplus \mathbb{C}\chi^0$. We don't get any higher cohomology as the compliment of a conical set in $\mathbb{R}$ only has nonzero reduced cohomology if it is the complement of the origin and no e_i^* is only at 0 above the function.

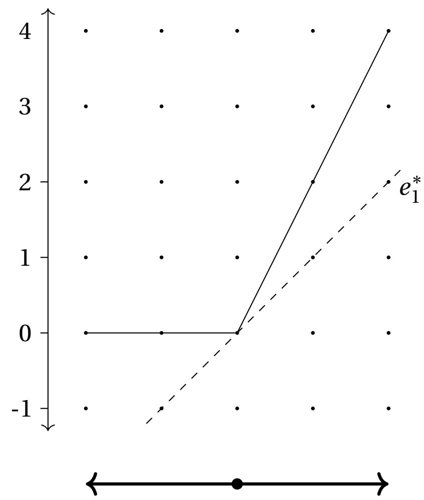


For $\mathcal{O}(-1)$ on the other hand we easily see that there neither is a e_i^* always on or above the function nor only at 0 not below so we get no cohomology.



For $\mathcal{O}(-2)$ we again get $H^0(\mathbb{P}^1, \mathcal{O}(D)) = 0$ but now e_1^* is only at 0 above or on the graph of the function giving us for $p > 1$

$$H^p(\mathbb{P}^1, \mathcal{O}(-2)) \cong \tilde{H}^{p-1}(\mathbb{R} \setminus \{0\}) \cong \begin{cases} \mathbb{C} & \text{for } p = 1 \\ 0 & \text{else} \end{cases}$$



Convex case

Corollary. *If X is affine, thus $|\Delta|$ a single cone and ψ_D linear, then $H^p(X, \mathcal{O}(D)) = 0$ for any D.*

Proof. Since for all $u \in M$ both $|\Delta|$ and $|\Delta| \setminus Z(u)$ are convex, thus contractible as ψ is linear on a single cone, the local cohomology groups in the exact sequence defining are sandwiched between 0, thus 0 themselves. □

As a generalization we get the following corollary.

Corollary. *If $|\Delta|$ is convex and $\mathcal{O}(D)$ is generated by sections then $H^p(X, \mathcal{O}(D)) = 0$ for all $p > 0$.*

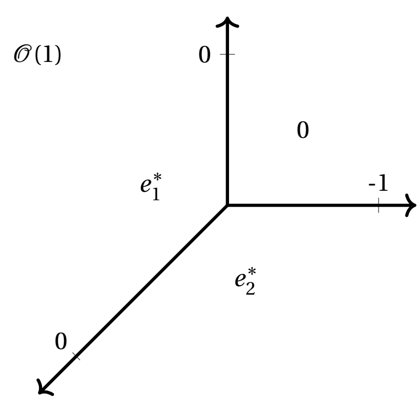
⚠In Fulton's convention convex function means that the set below the graph is convex ⚠

Proof. Follows as before as we saw that $\mathcal{O}(D)$ is generated by sections if and only if ψ_D is convex so $|\Delta| \setminus Z(u)$ is convex. □

Note that in $\mathbb{P}^1$ as in the examples before we immediately get for any $n \geq 0$ that $\mathcal{O}(-n)$ has trivial higher cohomology. Remember that being generated by sections means that for every $x \in X$ there exists a section, here a function to $\mathbb{A}^1$, that is not zero at x. For $n \leq 0$ we can just take constant functions so $H^p(X, \mathcal{O}(n)) = 0$ for $p > 0$ but since for $n < 0$ we have per definition only functions with a zero in our sheaf,thus the line bundle is not generated by it's sections so we can get higher cohomology.

Examples in $\mathbb{P}^2$

As before for $\mathcal{O}(1)$ we see that the associated function is convex so we have no higher cohomology. To find $H^0(\mathbb{P}^1, \mathcal{O}(1))$ we need to find P_D. Write $M = \{ae_1^* + be_2^* | a, b \in \mathbb{Z}\}$, then for $u \in P_D$ we need $a, b \geq 0$ for the first quadrant, but the left and lower cone demand $a, b \leq 0$ so $H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C}\chi^0$.



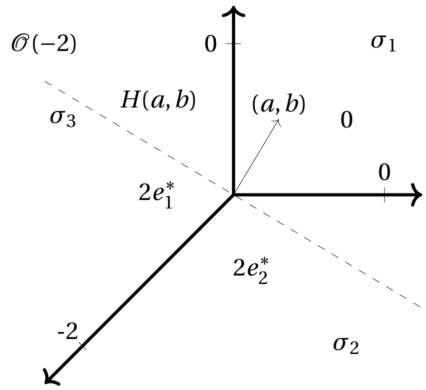
In $\mathbb{P}^1$ we got higher cohomology for $\mathcal{O}(-2)$. In $\mathbb{P}^2$ on the other hand we can verify that there is no cohomology. That is as $P_D = \emptyset$ as the set above ψ_D is a strictly convex cone, and for higher groups we first look at

$$Z(u) = \left\{v \in N \,\middle|\, \langle (u,v) \rangle \geq \psi_D(v)\right\} = \left\{ a,b \,\middle|\, av_1 + bv_1 \geq \begin{matrix} o & \text{on } \sigma_1 \\ -2v_1 & \text{on } \sigma_2 \\ -2v_2 & \text{on } \sigma_3 \end{matrix} \right\}$$

If we denote $H(a,b) = \{v \in M \,|\, \langle (a,b), v\rangle \geq 0\}$ we can write this as the set of the following intersections

$$Z(u) = (\sigma_1 \cap H(a,b)) \cup (\sigma_2 \cap H(a-2,b)) \cup (\sigma_3 \cap H(a,b-2))$$

Going through all case distinctions one can check that Z(u) is always simply connected and never isolates zero, thus there is no higher cohomology.



For $\mathcal{O}(-3)$ on the other hand one checks that we again get a nonzero first cohomology group at $u = 2e_1^* - e_2^*$ so

$$H^p(\mathbb{P}^2, \mathcal{O}(-3)) = \begin{cases} \mathbb{C}\chi^{2e_1^* - e_2^*} & \text{for } p = 1 \\ 0 & \text{else} \end{cases}$$

References

[Ful93] William Fulton. *Introduction to Toric Varieties. (AM-131)*. Princeton University Press, 1993. ISBN: 9780691000497.

[Hat02] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`.