A CRASH COURSE ON SHEAVES AND COHOMOLOGY

1. Sheaves

. We start with a topological space X – it could be a scheme, a toric variety, a manifold, or whatever topological space you like. From X, we may a *category* $Op(X)$, whose objects are the open sets $U \subset X$, and with a map $U \to V$ if $U \subset V$. If you don't know what a category is, you can think of $Op(X)$ as the set

$$\{U \subset X : U \text{ open}\}$$

which comes with a partial order, $U \leq V$ if $U \subset V$.

A pre-sheaf is a "contravariant functor" F from $Op(X)$ to the category of abelian groups. What this means is the following piece of data:

(1) For each $U \subset X$, an abelian group $F(U)$.
(2) For each $U \subset V$, a homomorphism

$$r_{UV} : F(V) \to F(U)$$

We will also assume $F(\emptyset) = 0$. The elements $s \in F(U)$ are called the *sections* of F along U.

A presheaf is called a sheaf if it satisfies the following "gluing" axiom:

- For any open sets $U, V \subset X$, set $W = U \cap V$. Then there is an exact sequence,

$$0 \longrightarrow F(U \cup V) \longrightarrow F(U) \oplus F(V) \xrightarrow{r_{UW} - r_{VW}} F(U \cap V)$$

What this says in words is that the sections of F on $U \cup V$ are exactly pairs of sections s_U on U and s_V on V which "glue", i.e. which agree on the overlap $U \cap V$.

Definition 1.1. The group $F(X)$ is called the group of *global sections* of F; it is also called the *0-th cohomology group* of F, denoted by $H^0(X, F)$.

From the sheaf axiom, it follows that we can compute $H^0(X, F)$ by writing $X = U \cup V$ and taking the kernel of the map

$$F(U) \oplus F(V) \to F(U \cap V)$$

Exercise 1.2. Let $\{U_1, \cdots, U_n\}$ be an open cover of X (i.e. $X = \cup_{i=1}^n U_i$). Put $U_{ij} = U_i \cap U_j$, and write r_{ij} and r_{ji} for the restriction maps[1]

$$r_{ij} : F(U_i) \to F(U_{ij})$$
$$r_{ji} : F(U_j) \to F(U_{ji})$$

Show that $F(X)$ is the kernel of the map

$$\oplus_{i=1}^n F(U_i) \xrightarrow{r_{ij} - r_{ji}} \oplus_{i<j} F(U_{ij})$$

[1]to avoid writing monstrosities such as $r_{U_i U_{ij}}$

(Hint: Use induction on n)

1.1. **Examples.** We now present some examples of sheaves.

Example 1.3. Let X be a manifold, and $\mathcal{O}_X$ be the sheaf of smooth functions on X, defined as
$$\mathcal{O}_X(U) = \{f : U \to \mathbf{R} : f \text{infinitely differentiable}\}$$
Restrictions $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for $U \subset V$ are literally restrictions of functions: $f : V \to \mathbf{R}$ restricts to $f|_U : U \to \mathbf{R}$.

Example 1.4. Let X be a manifold. We define Ω_X to be the sheaf of differential one-forms on X,
$$\Omega_X(U) = \{\omega \text{ a one form on } U\}$$
Restriction is as above. If U has a system of coordinates $x_1, \cdots, x_n$, differential forms are simply
$$\sum f_i dx_i$$
for functions f_i. But we do not assume that we can choose coordinates globally on U, and the definition still makes sense.

Example 1.5. Let X be an algebraic variety. The structure sheaf $\mathcal{O}_X$ is defined to be the set of maps
$$f : X \to \mathbb{A}^1$$
More explicitly, if $V \subset X$ is affine, isomorphic to $\operatorname{Spec} A$, we put
$$\mathcal{O}_X(V) = \{f : V \to \mathbb{A}^1\} = A$$
If $V \subset U$ is a basic open set, corresponding to a localization $A \to A_f$, the restriction map
$$\mathcal{O}_X(V) \to \mathcal{O}_X(U)$$
is exactly the localization
$$A \to A_f$$

To understand the example a little better, try:

Exercise 1.6. Let $X = \operatorname{Spec} A$ be an affine variety. Then
$$\operatorname{Hom}(X, \mathbb{A}^1) = A$$

Example 1.7. There are also variations to the structure sheaf above. We can put
$$\mathcal{O}_X^*(U) = \{f \in \mathcal{O}_X(U) : f \text{ is invertible}\}$$
and
$$\mathcal{K}_X^* = \{f \in K(U) - 0\}$$
Here $K(U)$ is the field of rational functions on U, and $*$ means we take the units in it, i.e. the non-zero elements.

We note that on a variety X, $\mathcal{K}_X^*$ has an amazing feature:
$$\mathcal{K}_X^*(U) = K(U)^* = K(X)^* = \mathcal{K}_X^*(X)$$
because the fraction field of X and any open set is the same. It follows that $\mathcal{K}_X^*$ is what's called a *constant sheaf*.

Example 1.8. A generalization of the above is as follows. Let X be a topological space, and G an abelian group. The constant sheaf $\underline{G}$ is defined by

$$\underline{G}(U) = \bigoplus_{\text{connected components of } U} G$$

The restriction $\underline{G}(V) \to \underline{G}(U)$ for $U \subset V$ is the "obvious one". A connected component U' of U is inside a unique connected component V' of V, and we take g in the component corresponding to V' to itself in the component corresponding to U'. Elements in connected components of V that do not meat U are mapped to 0.

Notice that $\mathcal{K}_X^*$ above is a constant sheaf with group $K^*(X)$. Note also that $\mathcal{K}_X^*$ is even more special than your average constant sheaf: because for a connected variety X, any open set is connected, $\mathcal{K}_X^*$ is is literally constant on any U.

Exercise 1.9. Show that the definition of constant sheaf we gave above is the "correct one", i.e. that the naive definition fails.

Specifically, let G be a group. Suppose F is the presheaf defined by $F(U) = G$ (and $F(\emptyset) = 0$). Find an example of a topological space X in which this rule does not give a sheaf.

1.2. **Cohomology.** Let X be a topological space, and F a sheaf on X. Cohomology groups take the idea of global sections a little further. We start with X and an open cover $U_1, \cdots, U_m$ of X. As we have seen above, we can compute $H^0(X, F)$ as the kernel of the map

$$\oplus F(U_i) \xrightarrow{r_{ij} - r_{ji}} \oplus_{i<j} F(U_{ij})$$

If we continue with this reasoning, we can form the triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$, quadruple intersections U_{ijkl}, and so on. To avoid making a total mess of the notation, for a subset $I = \{i_1, \cdots, i_n\}$ of $1, \cdots, m$, let's put

$$F(U_I) := F(U_{i_1} \cap U_{i_2} \cap \cdots U_{i_n})$$

We note that whenever $I \subset J$, we have a restriction map

$$F(U_I) \to F(U_J)$$

We now fix a set $J = \{i_1, \cdots, i_n\}$ of size n. We note that there are exactly n subsets of J of size $n-1$, namely the sets $J_k = \{i_1, \cdots, \widehat{i_k}, \cdots, i_n\}$ obtained by removing the k-th element i_k from J. For each k, we have a restriction map

$$r_{J,k} : F(U_{J_k}) \to F(U_J)$$

and so we can define a "differential"

$$d_J := \sum_{k=1}^{n} (-1)^{k-1} r_{J,k}$$

from

$$\bigoplus_{k=1}^{n} F(U_{J_k}) \to F(U_J)$$

Assembling these differentials together gives a map

$$\bigoplus_{I:|I|=n-1} F(U_I) \xrightarrow{d^{n-1}} \bigoplus_{J:|J|=n} F(U_J)$$

For the sake of concreteness, the sets of size 1 are precisely the singletons $i \in [1, \cdots, m]$, and the sets of size two are the pairs $i < j$. In this case, the map

$$\bigoplus_{I:|I|=1} F(U_I) \to \bigoplus_{J:|J|=2} F(U_J)$$

is exactly the map $r_{ij} - r_{ji}$ that we had above. Symbolically, we have a "complex"

$$0 \longrightarrow \oplus_{i=1}^n F(U_i) \xrightarrow{d_1} \oplus_{i<j} F(U_{ij}) \xrightarrow{d_2} \oplus_{|I|=3} F(U_I) \xrightarrow{d_3} \oplus_{|I|=4} F(U_I) \longrightarrow \cdots$$

We have seen that

$$H^0(X, F) = \ker d_1$$

Definition 1.10. The i-th cohomology group

$$H^i(X, F) = \ker d^{i+1} / \operatorname{Im} d^i$$

Actually the definition as given is incorrect. The correct definition requires that our cover $U_1, \cdots, U_n$ is "fine enough" (otherwise you could take X as your cover and deduce that $H^i(X, F) = 0$ always). However, we will not define what it means to be fine enough here. Just know that

- For manifolds, the U_i should be isomorphic to a ball in $\mathbb{R}^n$.
- For varieties, the U_i should consist of affine opens.
- For toric varieties $X = X(\Delta, N)$, we can take $X(\sigma, N)$ $(\sigma \in \Delta)$ as our cover.

Example 1.11. Let $X = S^1$ be the circle, and F the constant sheaf $\mathbb{Z}$. We will compute $H^i(X, F)$. We take a cover of S^1 by two intervals U_1, U_2 (for example, $(-\epsilon, \pi + \epsilon)$ and $(\pi - \epsilon, 2\pi + \epsilon)$ in polar coordinates). We see that $U_1 \cap U_2$ is the intersections of two smaller intervals, and so we have

$$F(U_1) = \mathbb{Z}, F(U_2) = \mathbb{Z}, F(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}$$

We compute the restriction

$$r_{12} : F(U_1) \to F(U_1 \cap U_2)$$

and see that it is simply the diagonal map

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

The retriction map $F(U_2) \to F(U_1 \cap U_2)$ is computed similarily. Thus, the map

$$d_1 : F(U_1) \oplus F(U_2) \to F(U_{12})$$

is the map

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

given by

$$(a, b) \mapsto (a - b, a - b)$$

As there is no third index, the map $d_2 : F(U_1 \cap U_2) \to \oplus_{I:|I|=3}$ is the 0 map. Thus, we find

$$H^0(X, F) = \ker d_i = \mathbb{Z}$$

and

$$H^1(X, F) = \ker d_2 / \operatorname{Im} d_1 = (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} = \mathbb{Z}$$

Exercise 1.12. Compute the cohomology groups of the sphere S^2 for the constant sheaf $\mathbb{Z}$.

Exercise 1.13. Let $X = \mathbb{P}^2$. Compute

$$H^i(X, \mathcal{O}_X)$$

(Hint: Use the open cover associated to the fan of $\mathbb{P}^2$ as a toric variety).

2. Long Exact Sequence

Calculations of cohomology groups can be very difficult. One of the most powerful tools we have to help us in this problem is the so called "long exact sequence" associated to a short exact sequence[2].

Theorem 2.1. *Let X be a topological space, and*

$$0 \to K \to F \to G \to 0$$

a short exact sequence of sheaves. Then we have a sequence

$$\cdots \longrightarrow H^{i-1}(X,G) \longrightarrow H^i(X,K) \longrightarrow H^i(X,F) \longrightarrow H^i(X,G) \longrightarrow H^{i+1}(X,K) \longrightarrow \cdots$$

which is "long exact", meaning that at each stage, we have "kernel = image": if we focus at any three term piece

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we have $\ker g = \operatorname{Im} f$.

Line Bundles and Divisors

3. Line Bundles and Divisors

In the lecture, we saw the definition of Cartier divisors, and we defined line bundles as equivalence classes of divisors up to rational equivalence,

$$\operatorname{Pic}(X) = \operatorname{CDiv}(X)/K^*(X)$$

With some cohomology under our belt, we are now in a position to give more intrinsic definitions of Cartier divisors and line bundles.

Let us begin with Cartier divisors. Remember that our definition was that a Cartier divisor on X was the data of

- An open cover $U_1, \cdots, U_n$ of X.
- A rational function f_i on U_i.
- A compatibility condition on U_{ij}:

$$\frac{f_i}{f_j} = u_{ij}$$

[2] If you've studied topology, this idea may be familiar to you, from the long exact sequence associated to the cohomology of a pair (X, A)

for a unit $u_{ij} \in \mathcal{O}_X^*(U_{ij})$ (a nowhere vanishing regular function). Remember we had a homomorphism to the group of Weil divisors by

$$(f_i) \to \operatorname{div}(f_i) = \sum_{V \text{ codimension } 1} \operatorname{ord}_V(f_i)[V]$$

– and the order did not depend on the choice of f_i precisely because of the compatibility on overlaps. How should we think of a Cartier divisor? There is two options.

Definition 3.1. Let X be a variety with fraction field K_X. A fractional ideal $I \subset \mathcal{K}_X$ is a subsheaf

$$I \subset \mathcal{K}_X$$

such that for each $g \in \mathcal{O}_X(U)$, and $f \in I$, we have

$$gf \in I$$

When should we consider two divisors (f_i) and (f_i') the same (for simplicity, let us assume they are defined on the same open cover of X)? Simply, when the f_i generate the same fractional ideal – when

$$\frac{f_i}{f_i'} = u_i$$

is a nowhere 0 regular function, or equivalently, when

$$\operatorname{div}(f_i) = \operatorname{div}(f_i')$$

This way, we see that we have for each i the data of an ideal

$$I(U_i) \subset K_X^*$$

defined by

$$I(U_i) = \mathcal{O}_X(U_i) f_i$$

But the compatibility on U_{ij} means that

$$I(U_i) = I(U_j)$$

and so we get all in all an ideal

$$I \subset K^*(X)$$

This ideal is locally principal: on U_i, it is generated by one element, namely f_i. We have thus arrived at

Definition 3.2 (1st definition)**.** A Cartier divisor is a locally principal fractional ideal $I \subset K^*$.

On the other hand, we have an alternative interpretation. The data of a Cartier divisor is equivalent to

$$f_i \in \mathcal{K}^*(U_i)/\mathcal{O}_X^*(U_i)$$

such that

$$\frac{f_i}{f_j} = 0 \in \mathcal{K}^*(U_{ij})/\mathcal{O}_X^*(U_{ij})$$

In other words, an element of $H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)$! To sum up, we have proven

Theorem 3.3. *We have*

$$\text{CDiv(X)} = \text{Fractional Ideals } I \subset K_X^* = H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)$$

4. Line Bundles

We can now move on to line bundles.

Definition 4.1. Let X be a variety. A line bundle L is a locally free sheaf of rank 1 on X.

We must now unpack this definition. A sheaf is free of rank 1 if it is isomorphic to the structure sheaf $\mathcal{O}_X$.

Exercise 4.2. Show that

$$\operatorname{Iso}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X^*(X)$$

To say that a line bundle is locally free means that we have an open cover $X = U_1 \cup \cdots \cup U_n$, and L is free on each U_i, $L|_{U_i} \cong \mathcal{O}_{U_i}$.

How can such an object be different from $\mathcal{O}_X$? After all, on each U_i we have an isomorphism

$$\phi_i : \mathcal{O}_X|_{U_i} \to L|_{U_i}$$

The point is that the isomorphisms may not be the same on U_{ij}. More precisely, if we restrict ϕ_i and ϕ_j to $U_{ij} = U_i \cap U_j$, their "difference"

$$\mathcal{O}_X|_{U_{ij}} = (\mathcal{O}_X|_{U_i})|_{U_{ij}} \xrightarrow{\phi_i|_{U_{ij}}} L|_{U_{ij}} \xrightarrow{\phi_j^{-1}|_{U_{ij}}} (\mathcal{O}_X|_{U_j})|_{U_{ij}} = \mathcal{O}_X|_{U_{ij}}$$

may not be the identity. By the exercise, the isomorphism

$$\phi_j^{-1}\phi_i|_{U_{ij}}$$

corresponds to a unique unit

$$u_{ij} \in \mathcal{O}_X^*(U_{ij})$$

Note that our units u_{ij} are not random. On U_{ijk} we have

$$\phi_j^{-1}\phi_i\phi_i^{-1}\phi_k\phi_k^{-1}\phi_j = 1$$

or, put otherwise,

$$u_{ij}u_{ik}^{-1}u_{jk} = 1$$

In other words, our line bundle is determined by an element $(u_{ij})_{i<j}$ in the kernel

$$\bigoplus_{i<j} \mathcal{O}_X^*(U_{ij}) \to \bigoplus_{i<j<k} \mathcal{O}_X^*(U_{ijk})$$

When should we consider two line bundles the same? A little bit of thought will convince you that we should do so if we have isomorphisms

$$\psi_i : L|_{U_i} \to L'|_{U_i}$$

such that

$$\psi_i = \psi_j$$

on U_{ij}. But we already have isomorphisms

$$\phi_i : \mathcal{O}_X|_{U_i} \to L|_{U_i}$$

and

$$\phi_i' : \mathcal{O}_X|_{U_i} \to L'|_{U_i}$$

and thus

$$(\phi_i')^{-1} \circ \psi_i \circ \phi_i$$

is an isomorphism

$$\mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$$

In other words,

$$\frac{\phi_i}{\phi_i'} = u_i \psi_i^{-1}$$

By the exercise, this is a unit $u_i \in \mathcal{O}_X^*(U_i)$. In that case, the elements

$$u_{ij} = \phi_j^{-1} \phi_i$$

and

$$u_{ij}' = (\phi_j')^{-1} \phi_i'$$

satisfy

$$\frac{u_{ij}}{u_{ij}'} = \frac{\phi_j^{-1} \phi_i}{(\phi_j')^{-1} \phi_i'} = \frac{u_i}{u_j}$$

Thus, isomorphic line bundles are exactly those for which the data u_{ij} differs by the image of

$$\bigoplus_i \mathcal{O}_X^*(U_i) \to \bigoplus_{i<j} \mathcal{O}_X^*(U_{ij})$$

Putting everything together, we see that the line bundles on X up to isomorphism are the kernel modulo the image in

$$\bigoplus_i \mathcal{O}_X^*(U_i) \longrightarrow \bigoplus_{i<j} \mathcal{O}_X^*(U_{ij}) \xrightarrow{d_1} \bigoplus_{i<j<k} \mathcal{O}_X^*(U_{ijk})$$

We have found

Theorem 4.3.

$$\text{Pic}(X) = \text{Line Bundles on } X/\text{Isomorphism} = H^1(X, \mathcal{O}_X^*)$$

5. Divisors to Line Bundles

In class, we defined

$$\text{Pic}(X) = \text{CDiv}/K_X^*$$

We have now seen also an interpretation

$$\text{CDiv}(X) = H^0(X, \mathcal{K}_X^*)$$

and

$$\text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$$

So how is this consistent with our definition? Simply the data of a divisor $(f_i) \in \bigoplus_i \mathcal{K}_X^*(U_i)$ maps to the element

$$u_{ij} = \frac{f_i}{f_j} \in \mathcal{O}_X^*(U_{ij})$$

This shows that there is a map

$$\text{CDiv} \to \text{Pic}$$

It is actually not so hard to see that the map has kernel $K^*(X)$. What is harder to see is that the map is surjective. For this, we employ the long exact sequence technique. We have a short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$$

The long exact sequence gives

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{K}_X^*)$$

Exercise 5.1. Let X be an algebraic variety, and F be the constant sheaf with values G. Show that

$$H^0(X, F) = G, H^i(X, F) = 0$$

for $i > 0$.

Applying this exercise to the constant sheaf $\mathcal{K}_X^*$, we get

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)/K_X^* = H^1(X, \mathcal{O}_X^*)$$

i.e. our isomorphism

$$\mathrm{Pic}(X) = \mathrm{CDiv}/K_X^*$$

from class. A common notation for the line bundle corresponding to a divisor D is

$$\mathcal{O}_X(D) = \{f \in \mathcal{K}_X | \mathrm{div} f \geq D\}$$

If $D = \sum a_i V_i$, this means that $\mathcal{O}_X(D)$ is the line bundle corresponding to the fractional ideal of rational functions which have at worse a pole of order a_i at V_i, no other poles.

Example 5.2. Let $X = \mathbb{P}^1$, $p = [0, 1]$. Let z be the coordinate in the chart $\{[x_0 : x_1] : x_1 \neq 0\}$. Then

$$\mathcal{O}_X(np) = \mathcal{O}_X \cdot \frac{1}{z^n}$$

is the line bundle corresponding to the fractional ideal of functions with a pole of order at worst n at p.

Example 5.3. Let $p = [0, 1]$ be as above, and q another point of $\mathbb{P}^1$. Show that

$$np \neq nq$$

as Cartier divisors, but

$$\mathcal{O}_X(np) \cong \mathcal{O}_X(nq)$$

as line bundles.

Exercise 5.4. Let $X = \mathbb{P}^1$, and U_0, U_1 the standard open cover $x_i \neq 0$. Let p be the point $[0, 1]$. Compute the element of

$$H^1(X, \mathcal{O}_X^*)$$

corresponding to the fractional ideal $\mathcal{O}(p)$.