TORIC VALUATIVE CRITERION

The notation will be: R is the discrete valuation ring, K its fraction field. A typical notation is $\operatorname{Spec} K = \eta$, $\operatorname{Spec} R = \{s, \eta\}$ – the spectrum of a dvr always has two points, a generic point η and a closed point s.

The reduction uses some preliminary facts:

(1) Any ring A can be considered as a monoid under multiplication, and any monoid S gives a ring $\mathbb{C}[S]$ as in class, and we have

$$\operatorname{Hom}(\operatorname{Spec} \mathbb{C}[S], A) = \operatorname{Hom}_{\mathrm{Mon}}(S, A)$$

(2) We can replace Y by any affine open that contains the special point of $\operatorname{Spec} R$. (Intuitively: we are checking if we can lift the map over the the special point, so what happens far from the point doesn't affect the lifting question).
(3) If $X \to Y$ is a map, and X is irreducible with an open dense U, when checking the valuative criterion we can take $\operatorname{Spec} K \to U \to X$. (Intuitively: we are checking if the fiber of $X \to Y$ has a "hole" over a point of Y. If such a hole were there, we could approach it from inside U).

For the valuative criterion, suppose we are given a toric morphism $X = X(F, N) \to Y = X(G, L)$. By point 2 above, we can assume $Y = X(\sigma, L)$ is affine, and by point (3) that $U = T_N$ is the dense open. Thus, we must check the lifting property for diagrams like this:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & T_N = X(0, N) \arrow[r] & X(F, N) \arrow[d] \\
\operatorname{Spec} R \arrow[rr] \arrow[urr, dashed] & & X(\sigma, L)
\end{tikzcd}$$

Likewise, the lifting property for one parameter subgroups corresponds to diagrams

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & T_N = X(0, N) \arrow[r] & X(F, N) \arrow[d] \\
\mathbb{A}^1 \arrow[rr] \arrow[urr, dashed] & & X(\sigma, L)
\end{tikzcd}$$

Let's call the first lifting LV, and the second LP (lifting valuation, lifting one parameter).

Lemma 0.1. *LP* $\iff$ *LV*

Proof. In either way, we will use property (1) to reduce the problems to commutative algebra about monoids. **Let's assume LV and show LP**.

Let

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & T_N = X(0,N) \arrow[r] & X(F,N) \arrow[d] \\
\mathbb{A}^1 \arrow[rr] & & X(\sigma, L)
\end{tikzcd}$$

be given. Take $R = \mathbb{C}[[t]]$, and $K = \mathbb{C}((t))$. Since $\mathbb{C}[t]$ is inside R, we have a map $\operatorname{Spec} R \to \mathbb{A}^1$, and similarily $\operatorname{Spec} K$ maps to $\mathbb{C}^*$. So we can extend our diagram to

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{C}^* \arrow[r] \arrow[d] & T_N = X(0,N) \arrow[r] & X(F,N) \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \mathbb{A}^1 \arrow[rr] & & X(\sigma, L)
\end{tikzcd}$$

By LV, we know we can fill the diagram as

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{C}^* \arrow[r] \arrow[d] & T_N = X(0,N) \arrow[r] & X(F,N) \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[urrr] & \mathbb{A}^1 \arrow[rr] & & X(\sigma, L)
\end{tikzcd}$$

uniquely. Suppose the image of the special point $s \in \operatorname{Spec} R$ lands in $X(\tau, N) \subset X(F, N)$ (since the affine toric varieties cover $X(F, N)$, it will always land in one of the $X(\tau, N)$). So we can reduce the diagram to

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{C}^* \arrow[r] \arrow[d] & T_N = X(0,N) \arrow[r] & X(\tau,N) \arrow[r, "\subset"] \arrow[d] & X(F,N) \arrow[dl] \\
\operatorname{Spec} R \arrow[r] \arrow[urrr] & \mathbb{A}^1 \arrow[rr] & & X(\sigma, L) &
\end{tikzcd}$$

To declutter a litter, let's focus on this piece of the diagram:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{C}^* \arrow[r] \arrow[d] & X(\tau,N) \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[urr] & \mathbb{A}^1 \arrow[r] & X(\sigma, L)
\end{tikzcd}$$

Since everything is affine, we have that this corresponds to a diagram of rings

$$\begin{tikzcd}
\mathbb{C}((t)) & \mathbb{C}[t, t^{-1}] \arrow[l] & \mathbb{C}[S_\tau] \arrow[l] \arrow[dll] \\
\mathbb{C}[[t]] \arrow[u] & \mathbb{C}[t] \arrow[l] \arrow[u] & \mathbb{C}[S_\sigma] \arrow[l] \arrow[u]
\end{tikzcd}$$

with $S_\tau = \tau^\vee \cap N^*$, $S_\sigma = \sigma^\vee \cap L^*$. By property (1), this translates to a diagram of monoids

$$\begin{tikzcd}
\mathbb{C}((t)) & \mathbb{C}[t, t^{-1}] \arrow[l] & S_\tau \arrow[l] \arrow[dll] \\
\mathbb{C}[[t]] \arrow[u] & \mathbb{C}[t] \arrow[l] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

Actually, we know a little more; since the maps from $\mathbb{C}^*, \mathbb{A}^1$ were assumed to be one parameter subgroups, the map from S_σ to $\mathbb{C}[t] = \mathbb{C}[\mathbb{N}]$ comes from a map $S_\sigma \to \mathbb{N}$, and so on. So we can further simplify to

$$\begin{tikzcd}
\mathbb{C}((t)) & \mathbb{Z} \arrow[l] & S_\tau \arrow[l] \arrow[dll] \\
\mathbb{C}[[t]] \arrow[u] & \mathbb{N} \arrow[l] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

We now take the canonical valuation $\mathrm{ord} : \mathbb{C}((t)) \to \mathbb{Z}$, which measures the order of a Laurent series (given

$$f(t) = \sum_{k \in \mathbb{Z}} c_k t^k$$

we have $\mathrm{ord} f(t)$ is the minimal k such that $c_k \neq 0$). So let's compose with the valuation. We get:

$$\begin{tikzcd}
\mathbb{Z} & \mathbb{C}((t)) \arrow[l, "\mathrm{ord}"'] & \mathbb{Z} \arrow[l] & S_\tau \arrow[l] \arrow[dll] \\
\mathbb{N} \arrow[u] & \mathbb{C}[[t]] \arrow[u] \arrow[l, "\mathrm{ord}"] & \mathbb{N} \arrow[l] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

But now note that the compositions $\mathbb{N} \to \mathbb{C}[[t]] \to \mathbb{N}$ and $\mathbb{Z} \to \mathbb{C}((t)) \to \mathbb{Z}$ are the identity. So the arrow $S_\tau \to \mathbb{C}[[t]] \to \mathbb{N}$ lifts to an arrow through $\mathbb{N}$, i.e. our diagram becomes

$$\begin{tikzcd}
\mathbb{C}((t)) & \mathbb{Z} \arrow[l] & S_\tau \arrow[l] \arrow[dl] \\
\mathbb{C}[[t]] \arrow[u] & \mathbb{N} \arrow[l] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

But from

$$\begin{tikzcd}
\mathbb{Z} & S_\tau \arrow[l] \arrow[dl] \\
\mathbb{N} \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

we can reverse our steps and build

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & X(\tau, N) \arrow[d] \\
\mathbb{A}^1 \arrow[r] \arrow[ur] & X(\sigma, L)
\end{tikzcd}$$

which is what we were trying to do. **So LV implies LP**.

Conversely, **assume LP** holds, and that we start with a diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & T_N = X(0, N) \arrow[r] & X(F, N) \arrow[d] \\
\operatorname{Spec} R \arrow[rr] \arrow[urr, dashed] & & X(\sigma, L)
\end{tikzcd}$$

where now R is an unnamed dvr (not necessarily $\mathbb{C}[[t]]$ as above). Our goal is to fill the dashed arrow. We first get the diagram of rings

$$\begin{tikzcd}
K & \mathbb{C}[N^*] \arrow[l] \\
R \arrow[u] & \mathbb{C}[S_\sigma] \arrow[l] \arrow[u]
\end{tikzcd}$$

for $S_\sigma = \sigma^\vee \cap L$. We use (1) now to reduce this to a diagram of monoids

$$\begin{tikzcd}
K & N^* \arrow[l] \\
R \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

We now compose with the valuation $\mathrm{ord} : K \to \mathbb{Z}$ (this is part of the data defining a valuation field and ring). We get

$$\begin{tikzcd}
\mathbb{Z} & K \arrow[l] & N^* \arrow[l] \\
\mathbb{N} \arrow[u] & R \arrow[l] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

which gives us a diagram of one parameter subgroups

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & T_N = X(0,N) \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \operatorname{Spec} R \arrow[r] & X(\sigma, L)
\end{tikzcd}$$

Since we are assuming LP, we know how to fill this diagram:

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & T_N = X(0,N) \arrow[r, "\subset"] \arrow[d] & X(F,N) \arrow[dl] \\
\mathbb{A}^1 \arrow[r] \arrow[urrr] & \operatorname{Spec} R \arrow[r] & X(\sigma, L) &
\end{tikzcd}$$

By the same argument as before, we can find an affine $X(\tau, N) \subset X(F, N)$ that contains the image of the point $0 \in \mathbb{A}^1$, and so we can simplify the picture a little:

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X(\tau, N) \arrow[d] \\
\mathbb{A}^1 \arrow[r] \arrow[urr] & \operatorname{Spec} R \arrow[r] & X(\sigma, L)
\end{tikzcd}$$

Now we will go back to algebra, to our diagram of monoids

$$\begin{tikzcd}
\mathbb{Z} & K \arrow[l, "\mathrm{ord}"'] & S_\tau \arrow[l] \arrow[dll] \\
\mathbb{N} \arrow[u] & R \arrow[l, "\mathrm{ord}"] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

and we use the definingproprety of a valuation ring: R is the subset of K consisting of elements f such that $\mathtt{ord}(f) \geqslant 0$. Since S_τ maps to K, and the composition $S_\tau \to K \to \mathbb{Z}$ factors through $\mathbb{N}$, the map $S_\tau \to K$ factors through R. Thus, we have

$$\begin{tikzcd}
\mathbb{Z} & K \arrow[l, "\mathrm{ord}"'] & S_\tau \arrow[l] \arrow[dl] \\
\mathbb{N} \arrow[u] & R \arrow[l, "\mathrm{ord}"] \arrow[u] & S_\sigma \arrow[l] \arrow[u]
\end{tikzcd}$$

and going back to geometry

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X(\tau, N) \subset X(F, N) \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \operatorname{Spec} R \arrow[r] \arrow[ur] & X(\sigma, L)
\end{tikzcd}$$

which is exactly what we wanted.

$\Diamond$