Toric maps and properness

Notes for my seminar talk.

1 Toric morphisms

Definition 1.1. *Let $\psi : N_1 \to N_2$ be a $\mathbb{Z}$-linear map and $\psi_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$ the induced map. It is called compatible with the fans $\triangle_1$ in $(N_1)_{\mathbb{R}}$ and $\triangle_2$ in $(N_2)_{\mathbb{R}}$ if for every cone $\sigma \subseteq X_{\triangle_1}$ there is a cone $\tau \subseteq X_{\triangle_2}$, such that $\psi_{\mathbb{R}}(\sigma) \subseteq \tau$.*

Definition 1.2. *Let $X_{\triangle_1}$, $X_{\triangle_2}$ be toric varieties with $\triangle_1$ a fan in $(N_1)_{\mathbb{R}}$ and $\triangle_2$ a fan in $(N_2)_{\mathbb{R}}$. A morphism $\phi : X_{\triangle_1} \to X_{\triangle_2}$ is* ***toric*** *if it is induced by a $\mathbb{Z}$-linear map $\overline{\psi} : N_1 \to N_2$ that is compatible with $\triangle_1$ and $\triangle_2$.*

Remark. *There is also an intrinsic definition of toric morphisms. A morphism $\phi : X_{\triangle_1} \to X_{\triangle_2}$ is toric if $\phi(T_{N_1}) \subseteq (T_{N_2})$ and $\phi|_{N_1}$ is a group homomorphism. You can show that every such morphism is induced by a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$ that is compatible with $\triangle_1$ and $\triangle_2$. We want to stress the combinatorial aspects of the toric morphisms here so we will use the first definition.*

Remark. *A toric morphism is equivariant, which means that it is compatible with the torus actions on the toric varieties.*

We will now illustrate the construction of a toric morphism starting from a $\mathbb{Z}$-linear map $\phi_{\mathbb{R}} : N_1 \longrightarrow N_2$ for cones. Note that this construction will glue to give a construction of toric morphisms between arbitrary toric varieties.

Let $N_1 = N_2 = \mathbb{Z}^2$ be lattices in $\mathbb{C}$ and $\phi_{\mathbb{R}} : N_1 \longrightarrow N_2$ be the identity map. Consider the cones σ generated by $\{e_1, e_2\}$ and τ generated by $\{2e_1 - e_2, e_2\}$.

Clearly, $\phi_{\mathbb{R}}$ induces a dual map $\phi_{\mathbb{R}}^* : N_2^* \longrightarrow N_1^*$ which in this particular case is still the identity. $\phi_{\mathbb{R}}^*$ also induces a map of semigroups $\tau^\wedge \cap N_2^* \longrightarrow \sigma^\wedge \cap N_1^*$. Here, $\tau^\wedge \cap N_2^*$ is generated by $\{e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*\}$ and $\sigma^\wedge \cap N_1^*$ is generated by $\{e_1^*, e_2^*\}$. Notice that this step is well-defined because σ maps into τ, so $\tau^\wedge$ maps into $\sigma^\wedge$.

This in turn induces an algebra-homomorphism

$$\phi^* : \mathbb{C}[x, y, z]/(y^2 - xz) \longrightarrow \mathbb{C}[x, y]$$
$$x \longmapsto x$$
$$y \longmapsto xy$$
$$z \longmapsto xy^2$$

So our morphism takes the form $\phi : \mathbb{C}^2 \longrightarrow V(y^2 - xz)$, $(s, t) \longmapsto (s, st, st^2)$. You can check that this defines a group homomorphism when restricted to the tori.

2 Properness

We will now turn to the discussion of when a toric morphism is proper. Note we have a classical notion of proper morphisms between varieties. In this paragraph we want to take a look at the combinatorial meaning of proper morphisms.

We start by stating the discrete valuative criterion for properness.

Theorem 2.1 (Discrete valuative criterion for properness)**.** *Let $f : X \longrightarrow Y$ be a finite type morphism of locally Noetherian schemes. Then f is proper if and only if for every commutative diagram*

$$\begin{array}{ccc} SpecK & \longrightarrow & X \\ \downarrow & & \downarrow f \\ SpecA & \longrightarrow & Y \end{array}$$

where A is a discrete valuation ring with fraction field K, there exists a unique lift $SpecA \longrightarrow x$ such that the following diagram commutes

$$\begin{array}{ccc} SpecK & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ SpecA & \longrightarrow & Y \end{array}$$

Remark. *Intuitively this theorem tells us that every curve in X with on point missing can be lifted to a curve with the point filled in.*

In the following we want to show the combinatorial meaning of properness. Indeed, proper toric morphisms are exactly the toric morphisms such that the preimage of every point in a cone in a fan is again in a cone in the given fan.

To show this property we will restrict ourselves to showing the criterion for one-parameter subgroups.

Definition 2.1. *Let $\triangle$ be a fan in $N_{\mathbb{R}}$ where N is a lattice. The support of $\triangle$ is defined to be*

$$|\triangle| = \bigcup_{\sigma \in \triangle} \sigma$$

Proposition 2.1. *Let $X_{\triangle_1}$, $X_{\triangle_2}$ be toric varieties with $\triangle_1$ a fan in $(N_1)_{\mathbb{R}}$ and $\triangle_2$ a fan in $(N_2)_{\mathbb{R}}$. A toric morphism $\phi : X_{\triangle_1} \to X_{\triangle_2}$ fulfills the valuative criterion for one-parameter subgroups λ^u, for all u such that $\phi(u) \in \triangle_2$*

$$\begin{array}{ccc} \mathbb{C}\backslash\{0\} & \xrightarrow{\lambda^u} & X_{\triangle_1} \\ \downarrow & & \downarrow f \\ \mathbb{C} & \xrightarrow{\lambda^{\phi(u)}} & X_{\triangle_2} \end{array}$$

if and only if $\phi^{-1}(|\triangle_2|) = |\triangle_1|$.

Proof. This is equivalent to the fact that for u such that $\phi(u) \in \triangle_2$,

$$\lim_{t \to 0} \lambda^u(t)$$

exists in $X_{\triangle_1}$ if and only if $u \in |\triangle_1|$. □

Theorem 2.2. *A toric morphism $\phi : X_{\triangle_1} \to X_{\triangle_2}$ is proper if and only if $\phi^{-1}(|\triangle_2|) = |\triangle_1|$.*

Proof. For a proof see Fulton's book on page 39 and 40. □

Remark. *We can also regard the toric varieties with the classical topology induced by the affine varieties that are glued together. Indeed, properness as a toric morphism corresponds to properness in the classical topology.*

Recall that a variety X is called complete when the constant morphism $X \longrightarrow \{pt\}$ is proper. Thus we get the following description of complete toric varietes.

Corollary 2.1. *A toric variety $X_{\triangle}$ where $\triangle$ is a fan in a lattice N is complete if and only if $|\triangle| = N_{\mathbb{R}}$, i.e. the fan covers the entire space.*

Remark. *We can again regard the classical topology on our toric variety. Then, completeness as a toric variety corresponds to compactness in the classical topology. For an example look at the gluing of $\mathbb{P}^1$ out of three cones in $\mathbb{C}$, the three cones cover all of $\mathbb{C}$ and $\mathbb{P}^1$ is compact.*