Projective Embeddings of Toric Varieties

Recall from previous talks that

T-Cartier Divisors on a toric variety $\simeq$ Piecewise linear functions on the cones.

Let $D = \{u(\sigma) \in M/M(\sigma)\}$ be a Cartier divisor of $X(\Delta)$. It defines a piecewise linear function on the support of the fan $|\Delta|$ whose restriction to each cone σ is given by $u(\sigma)$, i.e.

$$\psi_D(v) = \langle u(\sigma), v \rangle \quad \forall\, v \in \sigma.$$

If $D = \sum a_i D_i$, then ψ_D is given by

$$\psi_D(v_i) = -a_i \ .$$

Recall that the global sections of $\mathcal{O}(D)$ are given by

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{\operatorname{div}(\chi^m) \geq -D} \mathbb{C} \cdot \chi^m, \quad m \in M.$$

$$= \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

Now if $D = \sum a_i D_i$, then $m \in M$, $div(\chi^m) \geq -D$ is

$$\langle m, u_i \rangle \geq a_i \quad \forall i$$

These are just hyperplanes, so we have a polytope

$$P_D = \{ u \in M \mid \langle u, v_i \rangle \geq -a_i \quad \forall i \}$$

$$= \{ u \in M \mid u \geq \psi_D \text{ on } |\Delta| \}.$$

We now want to ask 2 questions:

(1) When is $\mathcal{O}(D)$ generated by its sections?; i.e. when is $H_0(X, \mathcal{O}(D))$ s.t for each point on X at least one global section is nonzero.

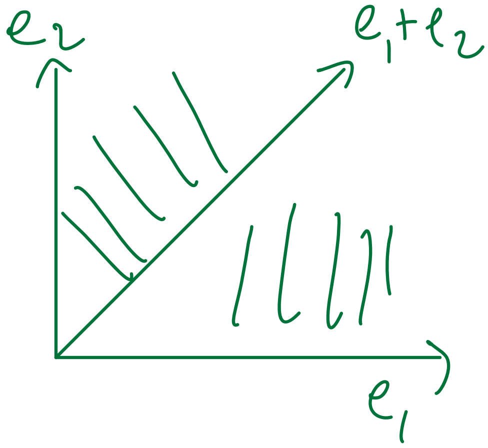
(2) When is the map

$$\Psi : X(\Delta) \longrightarrow \mathbb{P}^{r-1}$$
$$x \longmapsto (\chi^{u_1}(x), \dots, \chi^{u_r}(x)).$$

an embedding?

Clearly this map is only well-defined when $\mathcal{O}(D)$ is generated by sections (otherwise we would be mapping points to 0).

Note that if Δ is not complete, then P_D may not even be bounded. Consider

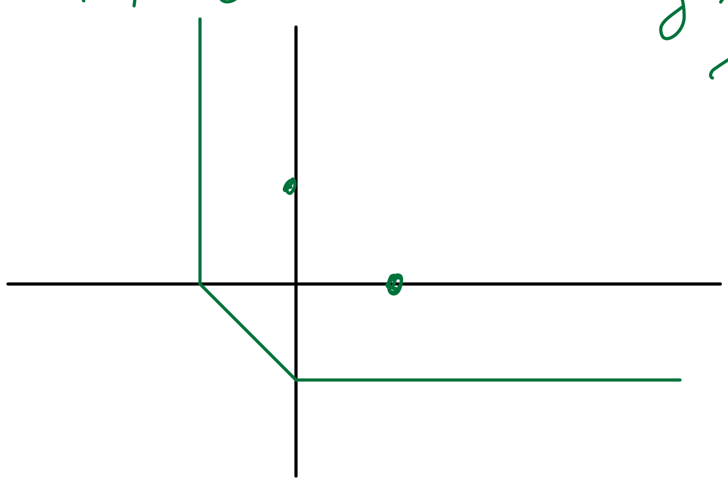


The Weil divisors are the codim 1 subvarieties, given by the 3 rays.
Consider the divisor

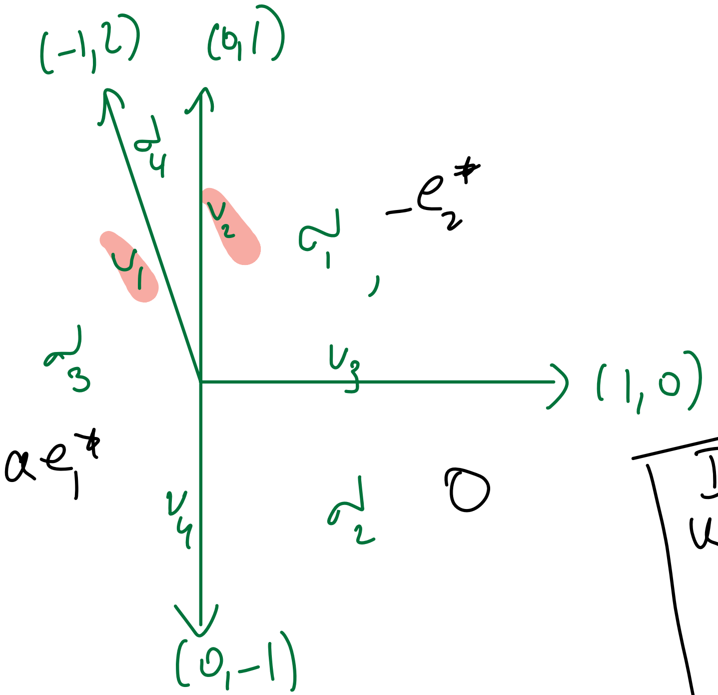
$$D = D_1 + D_2 + D_3 .$$

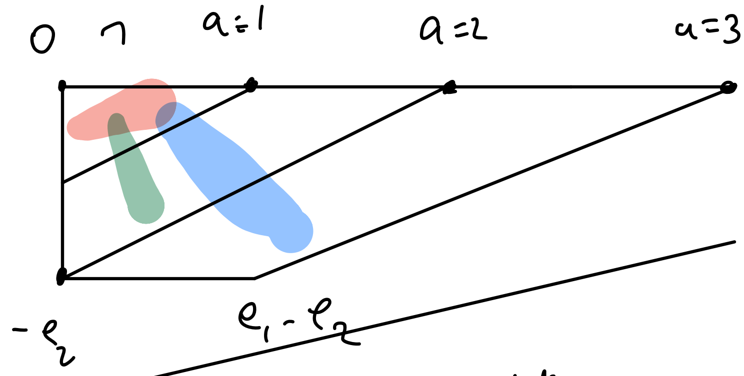
Then, $u \in P_D \iff$

$$\langle u, e_1 \rangle \geq -1 \iff x \geq -1$$
$$\langle u, e_2 \rangle \geq -1 \iff y \geq -1$$
$$\langle u, e_1+e_2 \rangle \geq -1 \iff x+y \geq -1$$



P_D not bounded!





If $a=1$, we can't even define $u(\alpha_4)$; since we would need $-x+2y=-1$ cannot have lattice coords.

$a=3$: $u(\alpha_4) = e_1^* - e_2^*$.

$D = aD_1 + D_2$. $u \in P_D$:

$\langle u, v_1 \rangle \geq -a \iff 2y - x \geq -a \iff y \geq \frac{x}{2} - \frac{a}{2}$

$\langle u, v_2 \rangle \geq -1 \iff y \geq -1$

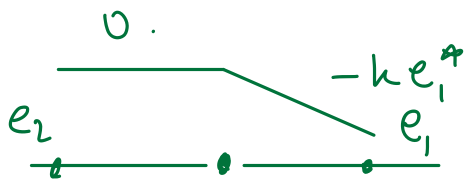
$\langle u, v_3 \rangle \geq 0 \iff x \geq 0$

$\langle u, v_4 \rangle \geq 0 \iff -y \geq 0 \iff y \leq 0$.

Def (Convexity). A real valued function ψ on a vector space is convex if

$$\psi(t \cdot v + (1-t)w) \geq t\psi(v) + (1-t)\psi(w).$$

Example. Consider the toric variety corresponding to $\mathbb{P}^1$, i.e.



$D = a_1 D_1 + a_2 D_2$, then

$$\psi_D(x) = \begin{cases} -a_1 x & x \geq 0 \\ -a_2 x & x \leq 0 \end{cases}$$

ψ_D is convex iff $a_1 + a_2 \geq 0$.

Prop Assume all maximal cones in Δ are n-dim. Let D be a Cartier divisor on $X(\Delta)$. Then $\mathcal{O}(D)$ is generated by sections iff ψ_D is convex

<u>Proof</u>. $\mathcal{O}(D)$ being generated by sections is equivalent to requiring the existence of $u(\sigma) \in M$ $\forall$ cones σ s.t

$$\text{(i) } \langle u(\sigma), v_i \rangle \geq -a_i \quad \forall i$$
$$\text{(ii) } \langle u(\sigma), v_i \rangle = -a_i \quad \forall\, v_i \in \sigma.$$

(i) tells us that $u(\sigma) \in P_D$. & we know P_D determines global sections

(ii) tells us that $\chi^{u(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ.

Now ψ_D is defined as the piecewise linear function whose restriction to each cone has the value in (ii). The fact that ψ_D is convex, means it fullfils (i) & hence is part of P_D & determines a global section.

Define

$$\varphi_D : X(\Delta) \longrightarrow \mathbb{P}^{r-1}$$
$$x \longmapsto \left(\chi^{u_1}(x) : \cdots : \chi^{u_r}(x)\right)$$

where $r = |P_D \cap M|$.

This map is well-defined as long as ψ_D is convex. Then $\mathcal{O}(D)$ is generated by global sections, so we can choose a basis.

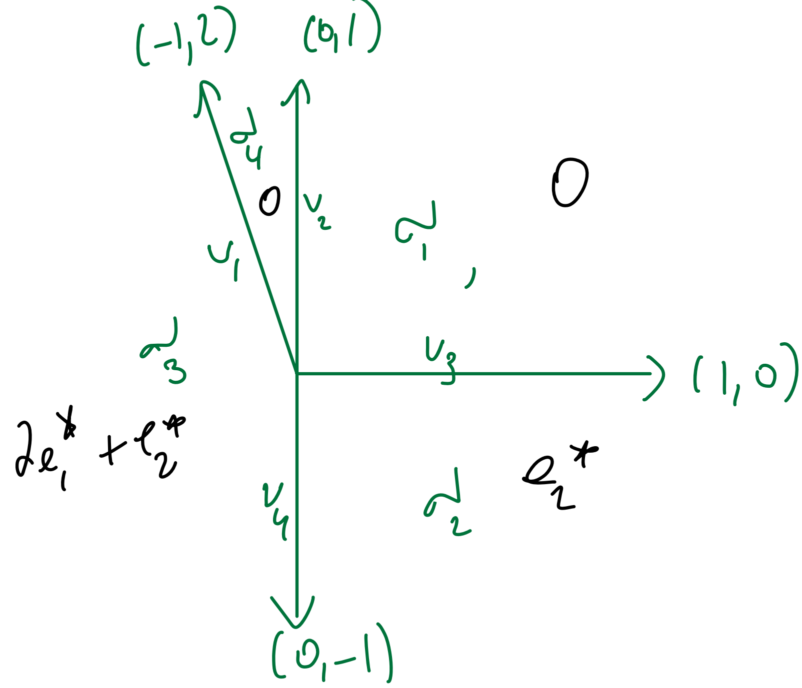
When is this map an embedding?

<u>Lemma</u> If $|\Delta| = N_{\mathbb{R}}$, the mapping φ_D is an embedding iff ψ_D is strictly convex & S_σ generated by $\{u - u(\sigma) : u \in P_D \cap M\}$ $\forall \sigma$.

<u>Proof sketch</u>. "$\Leftarrow$" ψ_D convex means the map $\varphi_D : X(\Delta) \to \mathbb{P}^{r-1}$ is well defined. Let σ be a n-dim cone & $u(\sigma) \in P_D^{\cap M}$ the corresponding function. ψ_D convex $\Rightarrow$ $\mathcal{O}_D$ is generated by $u(\sigma)$ on U_σ. The fact that ψ_D is strictly convex, yields that the inverse image by φ_D of $\mathbb{C}^{r-1} \subseteq \mathbb{P}^{r-1}$ where $u(\sigma) \neq 0$ is precisely U_σ, i.e. we reduce to the affine case.

Now Ψ_D restricted to an open $U_\sigma \to \mathbb{C}^{r-1}$ is simply $\chi^{u-u(\sigma)}$. But these $u-u(\sigma)$ generate S_σ, so the ring map is surjective implying embedding.

Consider the Hirzebruch surface.



ψ_D is convex but NOT strictly convex.

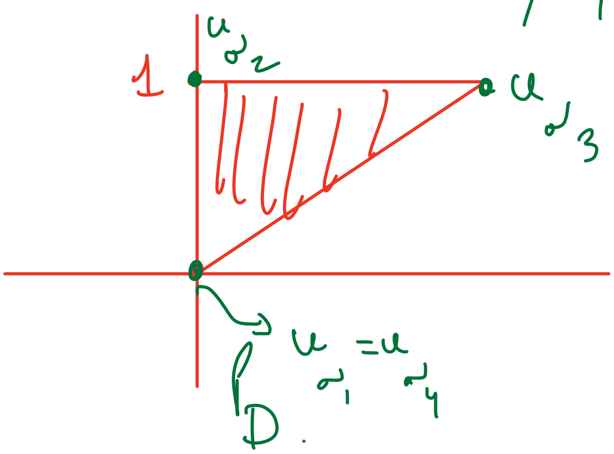
Consider $D = D_4$, $D' = D_2 + D_4$.

$u \in P_D$: $\langle u, e_2 \rangle \geq 0 \iff y \geq 0$

$\langle u, 2e_2 - e_1 \rangle \geq 0 \iff 2x - y \geq 0$

$\langle u, -e_2 \rangle \geq -1 \iff -y \geq -1 \iff y \leq 1$

$\langle u, e_1 \rangle \geq 0 \iff x \geq 0$.



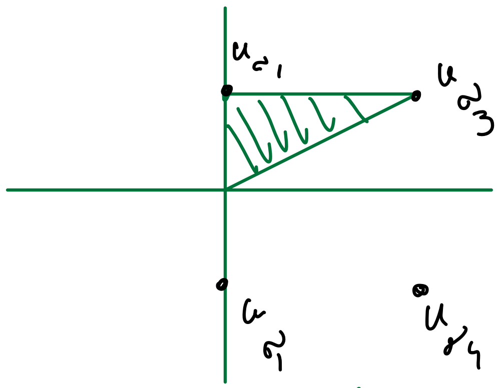
$\rightsquigarrow \mathcal{O}(D)$ is generated by sections

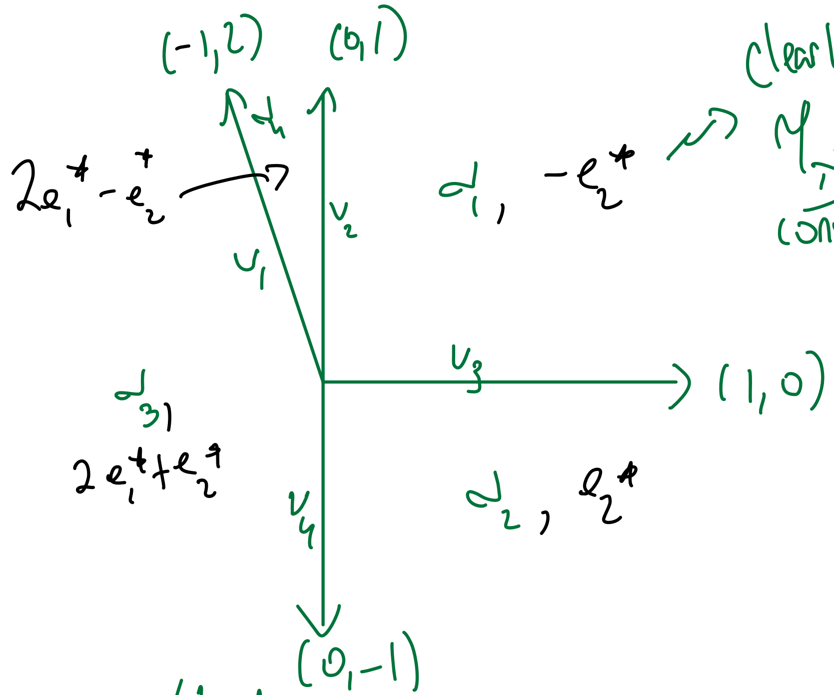
$u \in P_D'$: $\langle u, e_2 \rangle \geq -1 \iff y \geq -1$

$\langle u, 2e_2 - e_1 \rangle \geq 0 \iff 2y - x \geq 0 \iff y \geq x/2$

$\langle u, -e_2 \rangle \geq -1 \iff -y \geq -1 \iff y \leq 1$

$\langle u, e_1 \rangle \geq 0 \iff x \geq 0$.





$\rightsquigarrow$ clearly φ_D not convex.

$\mathcal{O}(D)$ not even generated by sections

Let's try and see this explicitly in coordinates:

$$\Psi_D : X(\Delta) \longrightarrow \mathbb{P}^2$$

$$x \longmapsto \left(1 : \chi^{e_2^*} : \chi^{e_2^* + 2e_1^*}\right)$$

$$\left(1 : Y(x) : X^2 Y(x)\right).$$

We have an issue at v_1 & σ_3 & σ_4. The "special point" ray $v_1 = (-1, 2)$ is mapped to $(1 : 0 : 1)$ but the special points on σ_3 & σ_4 go to $(0 : 0 : 1)$. This can't happen, since the orbit closure of v_1 is $\mathbb{P}^1$, and is mapped to $[a : 0 : b]$ in $\mathbb{P}^2$ while missing $[1 : 0 : 0]$. & Since maps from $\mathbb{P}^1$ are either const. or surj.

Special point of v_1 is the limit of the 1-para subgroup $(-1,2)$, i.e.

$$\lim_{t\to 0}(1,t^2,1) = [1:0:1].$$

Sp. pt. of σ_3: limit of 1-para subgrp $(-1,1)$.

$$\lim_{t\to 0}(1,t,t^{-1}) = [0:0:1]$$

Sp. pt. of σ_1 is the limit of $(-1,0)$

$$\lim_{t\to 0}(1,1,t^{-2}) = [0:0:1].$$

In summary:

Given a toric variety $X(\Delta)$ and a divisor D, we can define a PL function ψ_D.

If ψ_D is convex, then there is a well def. map $X \hookrightarrow \mathbb{P}^n$.

If ψ_D is strictly convex then this map is an embedding.