2 Goals: 1: How to get Δ, φ from ~~P~~ PCM

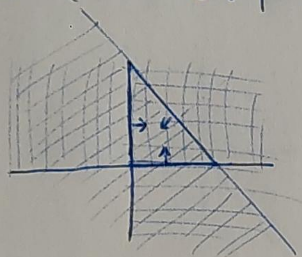
2: $P \to \Delta, \varphi$ and $\Delta, \varphi \to P$ are inverse constructions.

Formality:

A Polytope P is ~~the~~ convex hull of a finite set S, i.e. $P = \mathrm{Conv}(S)$.

Important: P can be written as $P = \bigcap_{i=1}^{S} H^+_{u_i, b_i}$

where $H^+_{u_i, b_i} = \{ v \in V \mid \langle u_i, v \rangle \geq b_i \}$ are ~~hyperplanes affine~~ halfspaces.

Picture 

$P = \mathrm{Conv}(0, e_1, e_2)$

$= H^+_{e_1, 0} \cap H^+_{e_2, 0} \cap H^+_{-e_1-e_2, -1}$

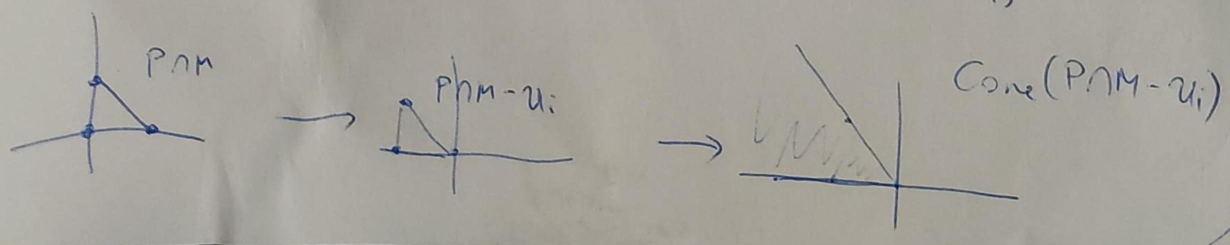
we write $H_{u,b} = \{ v \in V \mid \langle u, v \rangle = b \}$ for affine planes and facets of P are $F = \underbrace{\{ v \in P \mid \langle u, v \rangle = b \}}_{= P \cap H_{u,b}} \subset H_{u,b}$ given $P \subseteq H^+_{u,b}$

From this we can rewrite P into its facet presentation

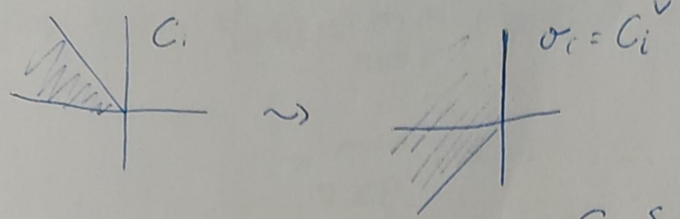
$$P = \{ v \in V \mid \langle u_F, v \rangle \geq -b_F,\ F \text{ facet } P \}$$

Getting Δ from P

for u_i vertex of P we obtain $C_i = \mathrm{Cone}(P \cap M - u_i)$



and then we set ~~σ~~ $\sigma_i = C_i^\vee$



we have that

$$C_i = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq 0 \ \forall\, u_i \in F \prec P \}$$

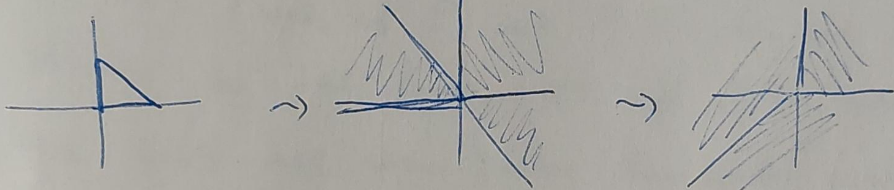
we generalise ~~$\sigma_Q = C_Q$~~

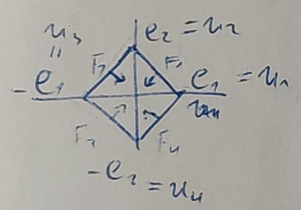
and then $\sigma_i = \mathrm{Cone}(u_F \mid u_i \in F \prec P)$

$= \{ v \in N_{\mathbb{R}} \mid \langle u_i, v \rangle \leq \langle j, v \rangle \}$

from this we generalise for $Q \prec P$ arb. face

$$\sigma_Q = \mathrm{Cone}(u_F \mid Q \text{ is in } F).$$





~~$\sigma_{e_1} = \mathrm{Cone}($~~

$F_1 := \{ m \in P \mid \langle -e_1 - e_2, m \rangle = -1 \}$

$F_2 := \{ m \in P \mid \langle e_1 - e_2, m \rangle = -1 \}$

$F_3 := \{ m \in P \mid \langle e_1 + e_2, m \rangle = -1 \}$

$F_4 := \{ m \in P \mid \langle -e_1 + e_2, m \rangle = -1 \}$

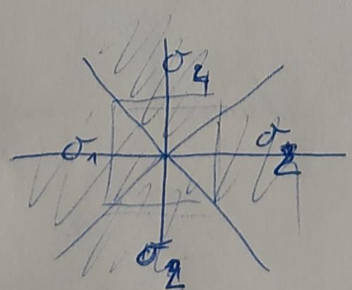
$\sigma_1 = \mathrm{cone}(-e_1 - e_2, -e_1 + e_2)$

$\sigma_2 = \mathrm{cone}(-e_1 - e_2, e_1 - e_2)$

$\sigma_3 = \mathrm{cone}(e_1 - e_2, e_1 + e_2)$

$\sigma_4 = \mathrm{cone}(e_1 + e_2, -e_1 + e_2)$

$\rightsquigarrow$



Getting φ from P

we set $\varphi(v) = \min_{u \in P} (u, v) \quad v \in N_{\mathbb{R}}$

$= \min_{u_i \in P} (u_i, v) \quad u_i$ vertex of P ← i will just state this, not insightful but useful

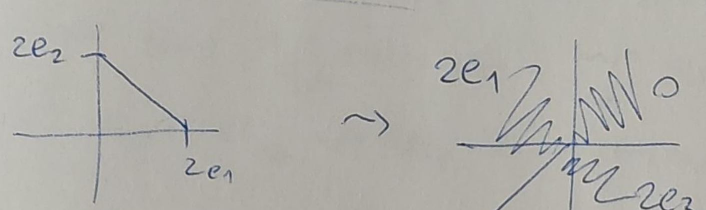
easy to recognise that ~~of~~ this indeed is a PL function

$\Rightarrow$ u_i is minimal on σ_i and u_i and u_j agree on $\sigma_j \cap \sigma_i$ by construction

reasoning: ~~P is conter~~ let u_i be a vertex. then u_i is contained in some Facets of P and P is contained in the halfspaces defined by the facets. $\Rightarrow$ all other vertices thus have strict ineq when taking the scalar product $\Rightarrow$ they cannot be minimal on the dual, but agree on boundries!

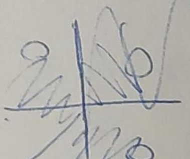
Remarks to G1

P gives a Pair:



still $\mathbb{P}^2$, different $\varphi \to D$

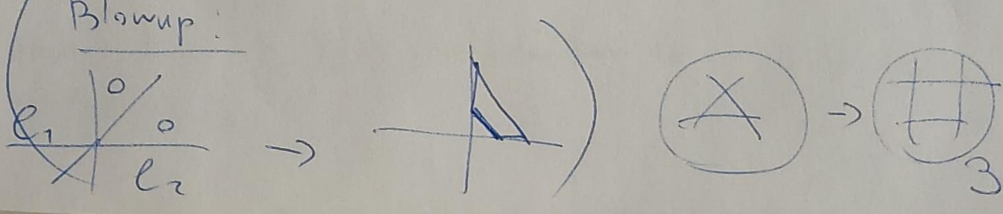
convex but not strict



→ the constant map $[x^i : x^i : x^i] = [1:1:1]$

Example on torn paper

Blowup:



Goal 2

we want to show that $P \rightsquigarrow \Delta_P, \varphi_P \rightsquigarrow P$ and

$\Delta, \varphi \rightsquigarrow P \rightsquigarrow \Delta_P \varphi_P$ are inverse.

First, $P_D = \{u \in M \mid \langle u, v\rangle \geq \langle u_\sigma, v\rangle \ \forall \sigma\}$ where $\varphi|_\sigma(v) = u_\sigma$ is how we obtain a polyhedron from Δ, φ.

it will be slightly easier if we can show that P_D has a slightly different form

let $u \in P$. then we have that $u - u_\sigma \in \sigma^\vee$ $\forall \sigma$ (definition of σ)

$\Rightarrow u \in \bigcap_{\dim(\sigma)=n} \sigma^\vee + u_\sigma$. it is easy to see that for $u \in \bigcap_\sigma \sigma^\vee + u_\sigma$ is also in P_D, thus we can write $P_D = \bigcap_\sigma \sigma^\vee + u_\sigma$

you can convince yourself that ~~$\bigcap_\sigma \sigma^\vee + u_\sigma = \mathrm{conv}(u_\sigma)$~~

$P_D \neq \mathrm{conv}(u_\sigma)$

let ~~P be a poly and~~ Δ_P, φ_P the fan and P_D the poly from Δ, φ. Let Δ_P, φ_P the Fan/funk generated from P. We can see that $\varphi_P|_\sigma(v) = u_\sigma = \min_{u_\sigma} (u_\sigma, v) = \varphi(v)$ ~~and~~ $\forall \sigma$ and that $\sigma_{u_\sigma} = \{v \in N_\mathbb{R} \mid (u_\sigma, v) \leq (u_{\sigma'}, v)\ \sigma' \neq \sigma\}$

$$= \{v \in N_\mathbb{R} \mid \varphi(v) = (u_\sigma, v)\} = \sigma$$

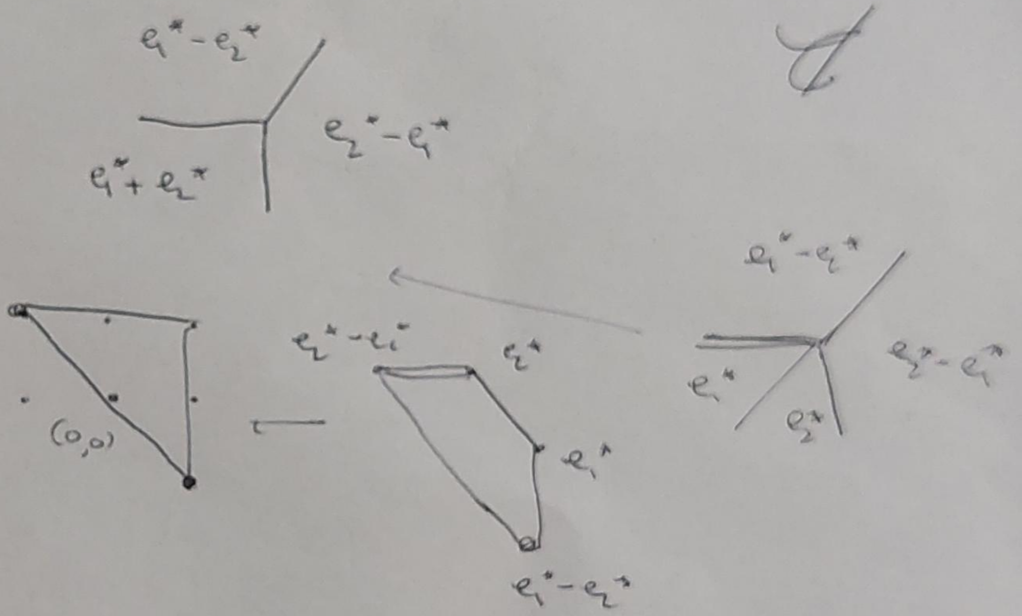
let P and the Δ_P, φ_P. let P_D be the poly from Δ_P, φ_P

It is easy to see $P_D \overset{\mathrm{conv}(u_\sigma)}{=} \subset P$ (Supp $P_D \not\subset P \Rightarrow \exists$ vertex $w \rightsquigarrow \Delta_P = \Delta_{P_D}$ $\rightsquigarrow \exists \sigma \in \Delta_P$ st $\sigma = \sigma_w \rightsquigarrow w \in P$ ↯)

$P \subset P_D$

let $u \in P$ be a vertex. then φ_P takes the value ~~on~~ u on some σ

$\Rightarrow u$ is a vertex of P_D


$e_1^* - e_2^*$
$e_2^* - e_1^*$
$e_1^* + e_2^*$
$e_1^* - e_2^*$
$e_2^* - e_1^*$
e_1^*
e_2^*
$e_2^* - e_1^*$
e_2^*
e_1^*
$e_1^* - e_2^*$
$(0,0)$