Cech Cohomology Examples

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1 Cohomology of Flasque Sheaves

Definition 1.1. A sheaf is said to be flasque if the restriction maps are surjective. We will use the following lemma about flasque sheaves.

Lemma 1.1. If $\mathcal{F}$ is a flasque sheaf then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Proof. See section 20.12 of the stacks project. □

One important class of flasque sheaves are the constant sheaves on an irreducible topological space. We define constant sheaves in general:

Definition 1.2. Let A be a ring and X a topological space. We will endow A with the discrete topology. The constant sheaf $\mathcal{A}$ on X is define by: $\mathcal{A}(U) = \{f : U \to A,\ f \text{ is a continuous function}\}$

Lemma 1.2. The constant sheaf on an irreducible topological space is flasque.

Proof. Let f be a continuous function from U to A. The image of f is the singleton because otherwise the preimage of two distinct points in the image would yield two nonempty open subsets that do not intersect. All non-empty open sets have nonempty intersection on an irreducible topological space. □

The above lemma shows us that for an irredicuble topological space we may just let $\mathcal{A}(U) = A$ with the restriction maps being the identity.

Another example of a flasque sheaf we will talk about is the skycraper sheaf.

Definition 1.3. Let X be an affine scheme. Fixed a closed point $x \in X$ and fix a set A (group, ring, etc). We define the skycraper sheaf on X by $A_x(U) = A$ if $x \in U$ and 0 otherwise.

Lemma 1.3. Let us consider the skyscraper sheaf over a point in $\mathbb{P}^1$ with values in the complex numbers. We claim that $H^1(\mathbb{P}^1, \mathbb{C}_p) = 0$.

Proof. Let us take $p = [0, 1]$. We may cover $\mathbb{P}^1$ with the affine opens $D(x_0) \cap D(x_1)$. The map on complexes is then: $\mathbb{C}_p(D(x_0)) = \mathbb{C}$ and $\mathbb{C}_p(D(x_1)) = 0$. We thus have that: $C^1(\mathcal{U}) = \mathbb{C} \oplus 0 \to C^2(\mathcal{U}) = 0$ this implies that the first and higher cohomology groups vanish.

□

2 Cohomology of Line Bundles on Projective Space

Definition 2.1. We define the line bundle sheaf on $\mathbb{P}^1$ by

$$\mathcal{O}(n)(U) = \{g/f : f \in k[x_0, \cdots, x_n]_e, g \in K[x_0, \cdots, x_n]_{e+n}, f(p) \neq 0\ \forall p \in U\}$$

We have the following facts:

1. $H^q(\mathbb{P}^1, \mathcal{O}(n)) = 0$ if $q \neq 0, 1$.
2. $H^0(\mathbb{P}^1, \mathcal{O}(n)) = \mathcal{O}(n)(\mathbb{P}^n)$.
3. $H^1(\mathbb{P}^1, \mathcal{O}(n)) = \frac{1}{x_0 x_1}\mathbb{C}[1/x_0, 1/x_1]_n$

Proof. A direct computation is easiest. We pick the affine cover $U = \{D_+(x_0), D_+(x_1)\}$. We observe that the $i \geq 1$ cohomology has to vanish. We also observe that the zeroth cohomology group is is just the global sections and in this case the global sections of $\mathcal{O}(n)$ are just the homogenous polynomials of degree n. So what we need to compute is the first cohomology group. Consider the sheaf $\mathcal{F} = \bigoplus_{n\in\mathbb{N}} \mathcal{O}(n)$. It is a fact that for Noetherian schemes cohomology commutes with taking direct sums. We will compute the cohomology for this sheaf and track the grading.

Proof. The zeroeth cohomology group has to be the global sections of the sheaf. This is a general fact for any sheaf F and cover $\mathcal{U}$, and follows because $C^0 = \prod \mathcal{O}(U_i)$ the $d_{ij}((s)_i) = s_{ij} - s_{ji} = 0 \implies s_{ij} = s_{ji}$ on intersections. By the glueing and uniqueness axioms, this corresponds to a global section.

We only need to figure out the first cohomology $H^i(\mathcal{U}, \mathcal{F})$, because the higher cohomology groups have to vanish.

In this case note that $\mathcal{F}(D(x_0)) = \mathbb{C}[x_0, x_1]_{x_0} = \mathbb{C}[x_0^{\pm 1}, x_1]$. Likewise, $\mathcal{F}(D(x_1)) = \mathbb{C}[x_0, x_1^{\pm 1}]$. We have also have $\mathcal{F}(D(x_0x_1) = \mathbb{C}[x_0^{\pm 1}, x_1^{\pm 1}]$

It is easy to see that $\mathbb{C}[x_0^{\pm 1}, x_1^{\pm 1}]/im(d)$ is just the quotient of two sets $A = \{x_0^{k_0}x_1^{k_1} : k_i \in \mathbb{Z}\}$ and $B = \{x_0^{m_0}x_1^{m_1} : m_0 \geq 0 \text{ or} m_1 \geq 0\}$. We may thus identify the quotient of A/B with the set of all monomial $x_0^{m_0}x_1^{m_1}$ with both $m_0 \leq 0$ and $m_1 \leq 0$. Thus we have that

$$H^1(X, \mathcal{F}) = \frac{1}{x_0x_1}\mathbb{C}[x_0^{-1}, x_1^{-1}]$$

By tracking the grading we recover the fact that:

$$\mathcal{O}(n) = \frac{1}{x_0x_1}\mathbb{C}[x_0^{-1}, x_1^{-1}]_n$$

□

3 Cartier Divisors and the Sheaf of Regular Functions

Fix for now an integral scheme. For an irreducible scheme the sheaf of meromorphic regular functions is the constant sheaf $K(X)$. We have the following exact sequence of sheaves:

$$0 \to \mathcal{O}_X^* \to K^* \to K^*/\mathcal{O}_X^* \to 0.$$

By the long exact sequence of cohomology we have the following long exact sequence:

$$0 \to \mathcal{O}_X^*(X) \to K^*(X) \to K^*/\mathcal{O}_X^*(X) \to H^1(X, \mathcal{O}_X^*) \to 0 = H^1(X, K^*) \to H^1(X, K^*/\mathcal{O}_X^*).$$

Here we are using the fact that $H^1(X, K^*) = 0$ since this sheaf is flasque. From this exact sequence we have that $Pic(X) = CDiv/K^*(X) \cong H^1(X, \mathcal{O}_X^*)$.

□

In order to get a more in depth and hands on understanding of the relationship between Cartier divisors and the picard group. We recall the following:

Definition 3.1. The picard group is the group of isomorphsim of classes of rank 1 $\mathcal{O}_X$ modules. It is a group under the operation $\mathcal{L}_1 \otimes \mathcal{L}_2$.

Definition 3.2. We denote the group $H^0(X, K^*/\mathcal{O}_X^*)$ to be the set of cartier divisors denoted $Div(X)$.

If $f \in H^0(X, K_X^*)$ then its image in $Div(X)$ is said to be a principal divisor. Two cartier divisors are said to be equivalent if their difference is a principal divisor.

We can represent a cartier divisor D by a system (U_i, f_i) where the U_i cover X and f_i is the quotient of two regular elements of $\mathcal{O}_X(U_i)$ and $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \implies f_i|_{U_i\cap U_j} \in f_j|_{U_i\cap U_j}\mathcal{O}(U_i \cap U_j)^*$.

So if we have two cartier divisors represented by $D_1 = \{(U_i, f_i)_i\}$ and $D_2 = \{(V_j, g_j)_j\}$ their product which we denote addively as D_1+D_2 is represented by $\{(U_i \cap V_j, f_i g_j)_{i,j}\}$. We will consider cartier divisors upto linear equivalence.

Given a cartier divisor we can construct a locally invertible sheaf $\mathcal{O}_X(D) \subseteq K_X$ defined by the system $\mathcal{O}_X(D)|_{U_i} = f_i^{-1}\mathcal{O}|_{U_i}$. Note that $D \geq 0 \iff \mathcal{O}(-D) \subseteq X$. [1]

[1] We say that $D \geq 0$ it can be represented by a system of $\{(U_i, f_i)_i\}$ with $f_i \in \mathcal{O}_X(U_i)$.