Def: ~~let $f: X \to Y$ be a function~~

Def: A morphism f between varieties X and Y is called <u>rational</u> if it maps $U \neq \emptyset \subset X$ open ~~ont~~ into Y. (U is dense in X)

Zarski topo

We call ~~such~~ f <u>birational</u>, if $\exists g: Y \to X$ rational with g is inverse to f.

In particular f is an iso between $U \subset X$ and $V \subset Y$.

$\Longrightarrow$ birat toric morph $\longrightarrow$ iso N, L)

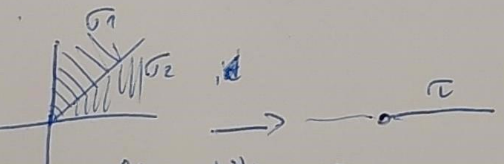
~~$X(N,F$~~ $X(F,N) \xrightarrow[\text{birat}]{\text{toric}} X(G,L) \Rightarrow T_N \cong T_L$

due to $Hom(T_N, T_L) = Hom(N, L)$

$\Rightarrow N \cong L$

(Idea: one-parameter subgroup $\cong N$ and φ birat $\bar{\varphi}$ iso)

Examples to get used to the notions of
- compatible
- proper
- birat

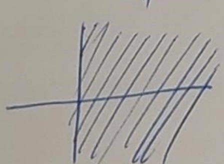


$\varphi((a,b)) \mapsto a$

is compatible
not proper
not birat

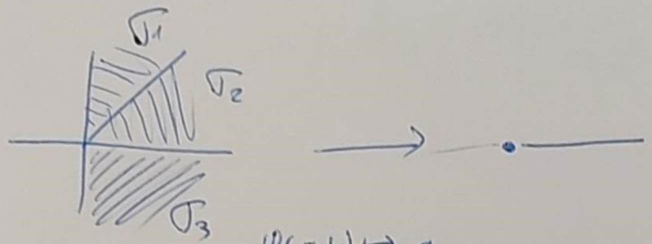
$\varphi^{-1}(\tau) = \{(a,b) \mid a \geq 0\}$

‖



$\Rightarrow \varphi^{-1}(\tau) \not\subseteq \bigcup_{\sigma_i \in \Delta} \sigma_i \Rightarrow$ not propper

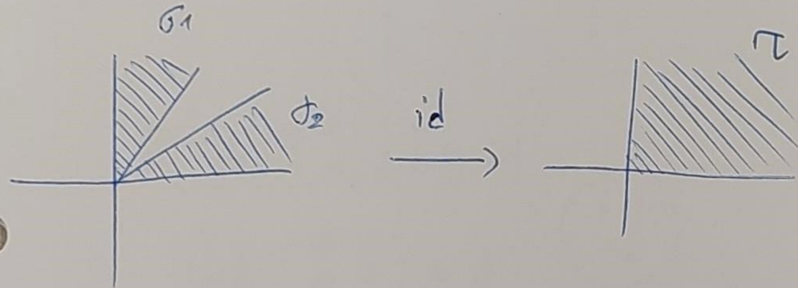
$\mathbb{Z}^2 \not\cong \mathbb{Z}$ thus not birat



is comp
is proper
not birat

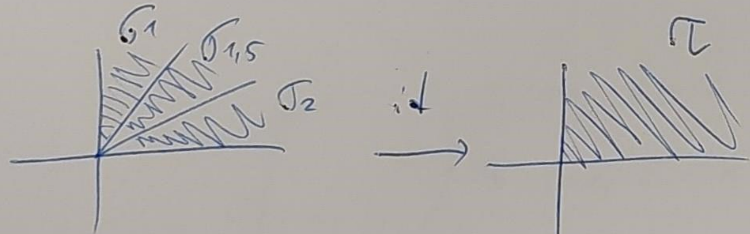
$\varphi(a,b) \mapsto a$

is propper du to "cancelling" the problem with our fan before.



is comp
is birat
not prop

$$id^{-1}(\mathbb{R}^2_{a\geq 0,\, b\geq 0}) = \mathbb{R}^2_{a\geq 0,\, b\geq 0} \neq \sigma_1 \cup \sigma_2$$



is all 3!

proper birat are exactly maps that "split up" cones

we call such Fans <u>refinements</u>

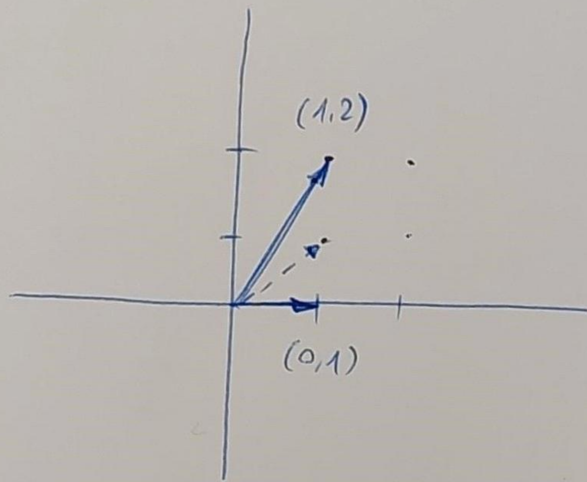
<u>Def</u>: Δ fan ~~of~~ in $N_{\mathbb{R}}$. A fan Δ' ~~re~~ is a refinement of Δ if every cone $\sigma' \in \Delta'$ is contained in some $\sigma \in \Delta$ and $|\Delta| = |\Delta'|$

why do we care about refinments?

they improve singularities in a very non-disruptive manner

now use cartoon pictures in order to show and give intuition

Ex 1: By refining we can get rid of sings and get non-sing toric var from sing tor var



$\det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2$

↳ singularity

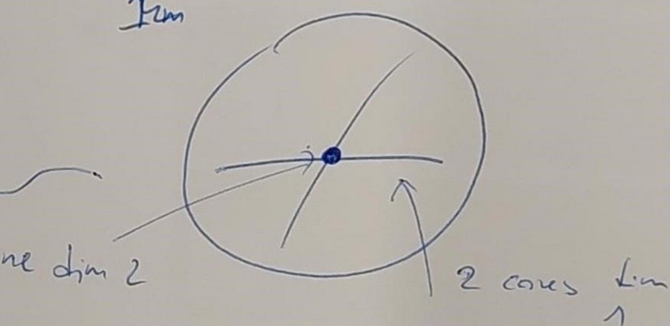
However adding $(1,1)$ we have 2 cones

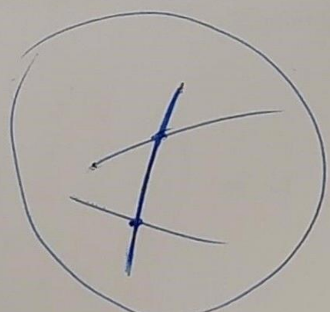
B) $\det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1$

$\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$

Cartoon:

Ifm





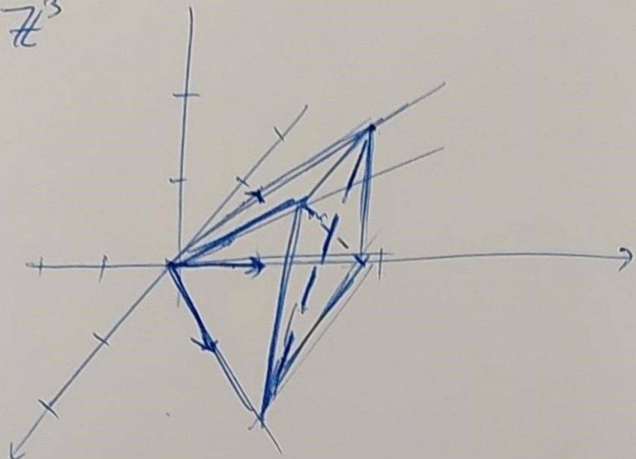
2 cones dim 2

3 cones dim 1

Important: Choice of $(1,1)$ mattered! if we choose poorly we do not make progress

Ex 2: $(1,0,0); (1,1,0); (1,0,1); (1,1,1)$

$\mathbb{Z}^3$

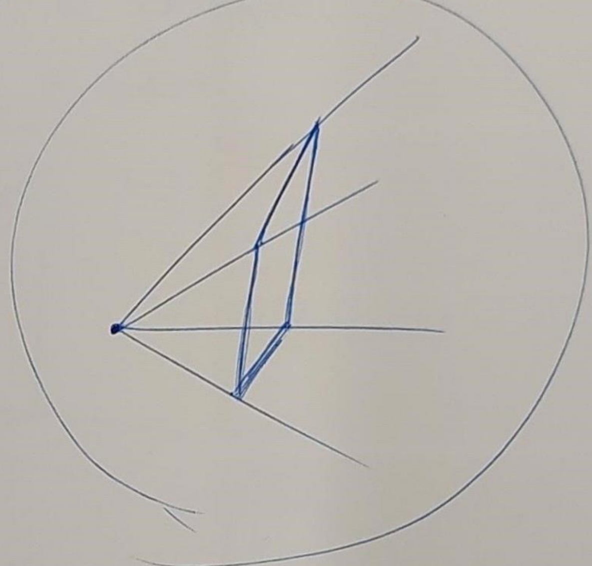


1 3 dim

4 2 dim

4 1 dim

→



by addin for example (...) gives

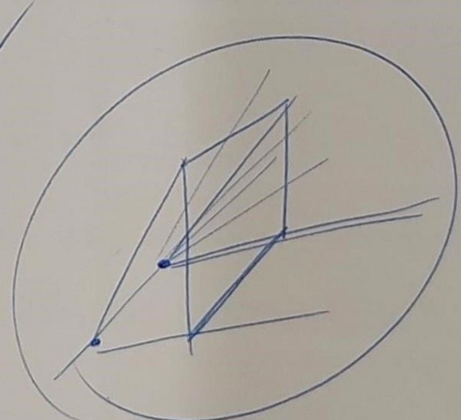
$$\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \checkmark$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1 - 1 - 1 = -1 \quad \checkmark$$

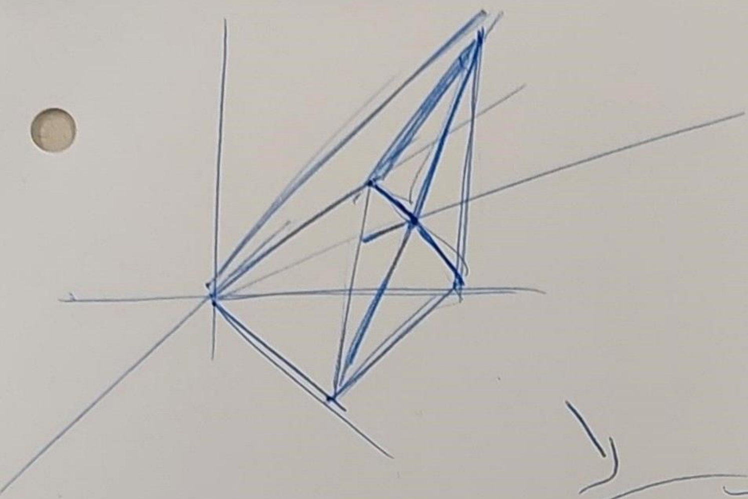
2 3 dim cones

5 2 dim

4 1 dim



Ex 2.2.

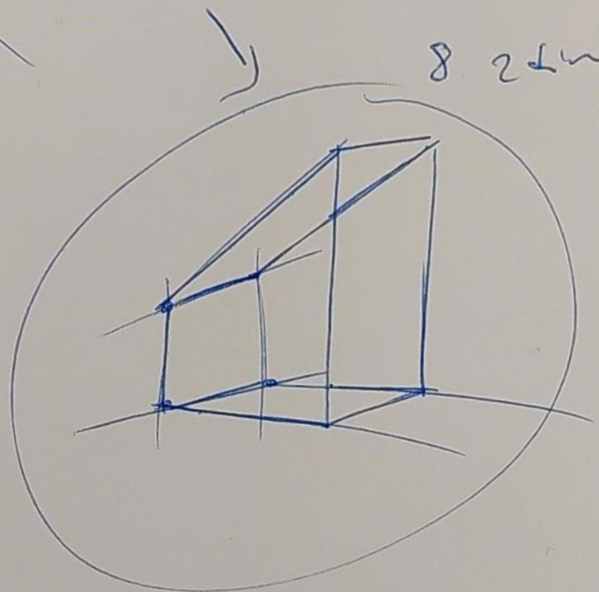


refinement along $\Sigma^2(\sigma)$ $(2,1,1)$

4 3dim cones

8 2dim

5 1dim



$$\det\begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix} = 1$$

$$\det\begin{pmatrix}1&1&2\\0&0&1\\0&1&1\end{pmatrix} = -1$$

$$\det\begin{pmatrix}1&1&2\\1&1&1\\0&1&1\end{pmatrix} = 1+2-1-1 = 1$$

$$\det\begin{pmatrix}1&1&2\\0&1&1\\1&1&1\end{pmatrix} = 1+1-2-1 = -1$$

important def:

Star subdivision along $v \in \sigma$ is

$\mathrm{Star}(v) = \{\mathrm{conv}(\tau, v) \mid \tau < \sigma \text{ is a face}\}$

look at Ex 1, 2, 2.2.

note that diff authors mean different things 2.2. in CLS