Smoothness of Toric Varieties

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Abstract

This paper presents the notes for the seminar in "Toric Geometry" of the 17.10.2023. We will discuss smoothness of Toric Varieties.

1 Smoothness of General Varieties

At first, lets start with a short repetition of relevant definitions.

Definition 1.1. *Let* $X = \mathbb{K}[x_1, ..., x_n]/(f_1, ..., f_n)$ *be a variety over the field* $\mathbb{K}$*,* $x \in X$*. The tangent space* $T_{X,x}$ *is given by*

$$T_{X,x} = \{Derivations\, D : X \to K\, at\, x\}$$

where Derivations are $\mathbb{C}-$*linear maps that satisfy the Leibnitz rule.*

Theorem 1.2.

$$T_{X,x} \simeq Hom(m/m^2, \mathbb{K})$$

with $m = \{g|g(x) = 0\} \subset X$ *being the maximal ideal at this point.*

Proof. The isomorphism is given by

$$D \mapsto \phi_D(f) = D(f)$$

with its inverse

$$\phi \mapsto D_\phi(f) = \phi(f - f(x))$$

□

Hence, $(m/m^2)^v \simeq T_{X,x}$ and we can define the following:

Definition 1.3. *The Zariski-cotangent space* $\Omega^{Zar}_{X,x}$ *at the point x is given by*

$$\Omega^{Zar}_{X,x} = m/m^2$$

Now, we can continue with the definition of smoothness.

Definition 1.4. *Let* X *be a variety,* $x \in X$*.* X *is smooth at* x *if*

$$dim(\Omega^{Zar}_{X,x}) = dim(X)$$

There exists a useful and well-known criterion that helps us the determine whether a variety is smooth at a point x or not.

Proposition 1.5. *Let* $X = \mathbb{K}[x_1, ..., x_n]/(f_1, ..., f_n)$ *be a variety.* X *is smooth at a point* $x \in X$ *if*

$$rank(\frac{\delta f_i}{\delta x_j}(x)) = n - dim(X)$$

Differently said, the Jacobian has to have maximal rank.

Proof. Let n be the ideal of x in $\mathbb{A}^n$. Write $I = (f_1, ..., f_n)$ for the ideal of X. Taking the quotient map $k[x_1, ..., x_n] \to I$, we see there is a map

$$n/n^2 \to m/m^2$$

i.e. a map $\Omega^Z_{\mathbb{A}^n,x} \to \Omega^Z_{X,x}$. It is easy to see that $\Omega^Z_{\mathbb{A}^n,x}$ is n-dimensional, generated by the functions $x_i - a_i$, where $x = (a_1, ..., a_n)$. These correspond to the Kahler differentials dx_i. We have

$$m/m^2 = (n/I)/(n^2 + I/I) = n/(n^2 + I)$$

and so the kernel of the map $n/n^2 \to m/m^2$ is precisely I. Under our isomorphism above, an element $f_i \in I \subset N$ goes to the element

$$df_i$$

Observe, that by Taylor's expansion it holds that

$$df_i = \sum \frac{\delta f_i}{\delta x_j}(x) dx_j$$

Thus, we see that $\Omega^Z_{X,x}$ is k-dimensional if and only if the space generated by the df_i has dimension $n - k$. This is exactly the rank of the Jacobian. $\square$

Example 1.6. *Consider the variety $X = \mathbb{K}[x, y, z]/(xz = y^2)$. The Jacobian is given by $(z \quad -2y \quad x)$, so the Jacobian has maximal rank if and only if $(x, y, z) \neq 0$. Remark that the corresponding fan has the rays $(1, 0), (1, 2)$.*

2 Smoothness of Toric Varieties

In case of Toric Varieties, there exists a theorem that connects the smoothness of $X(\sigma, N) = Spec(\mathbb{C}[S_\sigma])$ to a property of the cone σ. But first, we continue with a useful proposition.

Proposition 2.1. *If $\sigma \subset N_\mathbb{R}$ and $span(\sigma) \subsetneq N_\mathbb{R}$ Then there exist two lattices $N_1, N_2 \subset N$, such that*

$$N = N' \oplus N''$$

with $N' = span(\sigma) \cap N$. This implies that

$$X(\sigma, N) = X(\sigma, N') \times T_{N''}$$

Proof. Note that N/N_1 is torsions-free. By the classification theorem for free modules over PID's N/N_1 is free. Let $B \subset N$ denote the elements that map to a basis of N/N_1 under the projection. We set $N_2 = \{\mathbb{Z} - linear\, combinations\, in\, B\}$.
We condlude by noting that $N_1 \times N_2 \to N, (n_1, n_2) \mapsto n_1 + n_2$ is bijective. It has trivial kernel since $n_1 + n_2 = 0$ implies $n_2 \in N_1$, so by arguing via the basis, $n_2 = 0$, so n_1 is also 0. If $y \in N$, then again arguing via B, there in $n_2 \in N_2$ with $y - n_2 \in N_1$, from which bijectivity follows.
Remark, that the implication for the varieties follows from the duality between sums of algebras and Products of their varieties. Further note, if $N^v = M$, then

$$Spec(\mathbb{C}[M]) = T_N$$

$\square$

Remark 2.2. *Note, that if $X(\sigma, N) = X(\sigma, N') \times T_{N''}$ holds, then $X(\sigma, N)$ is smooth if and only if $X(\sigma, N')$ is smooth.*

Definition 2.3. *A cone $\sigma \subset N$ is full dimensional if $Span(\sigma) = N$*

Note, that this proposition allows us the following: If we want to show smoothness of a toric variety $X(\sigma, N)$, then we can assume that σ is full dimensional.
If this would not be the case, one can set $N' = span(\sigma) \cap N$. Then prove that $X(\sigma, N')$ is smooth which is by the proposition equivalent that $X(\sigma, N)$ is smooth.

Now, we can state the main theorem of todays lesson:

Theorem 2.4. *A toric variety $X(\sigma, N) = Spec(\mathbb{C}[S_\sigma])$ is smooth if and only if σ is generated by a part of a basis for N.*

Proof. Let $n = dim X$. Note, we now can assume that σ is full dimensional, so σ is generated by $(e_1, ..., e_n)$.
In this case, the toric action has a unique fixed point x_σ which is the maximal ideal $m = \langle \chi^m | m \in S_\sigma \setminus \{0\} \rangle \subset \mathbb{C}[S_\sigma]$. Observe that $m^2 = \langle \chi^w | w = u + v, u \neq 0, v \neq 0 \rangle$.
Define $B = \{\sum_{i=1}^n a_i e_i^* | a_i \in [0,1]\} \subset \sigma^v \cap M$
Remember, X is smooth at x_σ if

$$dim(\Omega^{Zar}_{X,x_\sigma}) := dim(m/m^2) = dim(X)$$

Now, one can argue that if $u \notin B \subset M$, then $\chi^u \in m^2$, because then $u = \sum u_i$ for some $u_i \in B$.
This means that m/m^2 only contains the elements in B.
As σ is full dimensional, B already contains at least the n generators $e_1^*, ..., e_n^*$.
Hence, smoothness of X at x_σ is equivalent to B containing no other points that the n generators which is equivalent to that $(e_1^*, ..., e_n^*)$ is part of a basis for M which is equivalent to that $(e_1, ..., e_n)$ is part of a basis for N. □

Remark 2.5. *Be aware that being a basis of N is a stronger statement that being a basis for $N_\mathbb{R}$. Consider for example the cone generated by $(1,0),(1,2)$. This is a basis for $N_\mathbb{R}$ but not for N, because you can consider*

$$(1,1) = \frac{1}{2}(1,0) + \frac{1}{2}(1,2).$$

Remark 2.6. *Be aware that this has a very geometric intuition.*

Remark 2.7. *To check whether a cone σ is part of a basis there exists a handy criterion. Any basis is equivalent to any other basis by a matrix, which is invertible. Hence, we can start with the standard basis and interpret $A = (v_1, ..., v_n)$ as this matrix.*
If this matrix A is invertible, then $(v_1, ..., v_n)$ is a basis of $N_\mathbb{R}$. To be a basis for N, we have the stronger condition

$$det(v_1, ..., v_n) = \pm 1$$

as ± 1 are the only units in $\mathbb{Z}$.

Example 2.8. *Consider $X = Spec(\mathbb{K}[x,y,z]/(xz = y^2)$ from above. The fan of it is given by the single cone with generators $(1,0),(1,2)$. We have seen above that these are no basis which implies by the theorem that X is not smooth which is exactly what we figured out before.*

Example 2.9. *If a cone is generated by only one element, then the corresponding variety always is smooth.*

Example 2.10. *For every non-simplicial cone $\sigma \subset N$ the toric variety $X(\sigma, N)$ is singular. The reason is that non-simplicial cones have more generators than the dimension of N.*

References

[Cox10] David Cox. *Toric Varieties.* 2010.

[Ful93] William Fulton. *Introduction to Toric Varieties.* 1993.