NONSINGULAR VARIETIES

Let $X = \operatorname{Spec} A = k[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$ be an affine variety. In other words, X is the set of points

$$\{(x_1, \cdots, x_n) : f_i(x_1, \cdots, x_n) = 0, i = 1, \cdots, m\}$$

We will work around a point $x \in X$. We will write $\mathfrak{m}$ for the maximal ideal of functions $f \in A$ such that $f(x) = 0$.

Recall from differential geometry:

Definition 1. A derivation $D : A \to \mathbb{C}$ at x is a $\mathbb{C}$-linear map D that satisfies the Leibniz rule:

$$D(fg) = f(x)D(g) + g(x)D(f)$$

The tangent space $T_{X,x}$ of X at x is the set of all derivations $A \to \mathbb{C}$ at x.

Definition 2. The cotangent space $\Omega_{X,x}$ at x is the vector space generated by elements df, $f \in A$, with the relations

$$\begin{aligned} d(cf) &= cd(f) \text{ for all constants } c \\ d(f+g) &= d(f) + d(g) \\ d(fg) &= f(x)dg + g(x)df \end{aligned}$$

In other words,

$$\Omega_{X,x} = \langle df, f \in A \rangle / d(fg) = f(x)dg + g(x)df$$

Remark 0.1. We have $d(1) = 0$ since $d(1) = d(1 \cdot 1) = 1d(1) + 1d(1) = 2d(1)$. Thus, for every constant c, we have $d(c) = cd(1) = 0$. (This says, the derivative of a constant is 0).

Proposition 0.2. *The tangent and cotangent space are dual vector spaces.*

Proof. Let D be a derivation at x. We define a map $\Omega_{X,x} \to \mathbb{C}$ by $D(df) = D(f)$. We note that this is well defined, as $D(d(fg)) = D(fg) = f(x)D(g) + g(x)D(f)$ and $D(f(x)dg + g(x)df)$ is the same. So we get a map $T_{X,x} \to \operatorname{Hom}(\Omega_{X,x}, \mathbb{C})$. We claim the map is injective. This is easy: if $D(df) = 0$ for all f, then $D = 0$. Conversely, suppose $\phi : \Omega_{X,x} \to \mathbb{C}$ is a homomorphism. We define a derivation by $D(f) = \phi(df)$. Again, this is well defined since $D(fg) = \phi(dfg) = \phi(f(x)dg + g(x)df) = f(x)D(g) + g(x)D(g)$, and it is clear that ϕ is the image of D under the map defined above. So the map is also surjective, and so an isomorphism. ◊

Definition 3. The Zariski cotangent space of X at x is the vector space

$$\Omega^Z_{X,x} := \mathfrak{m}/\mathfrak{m}^2$$

The Zariski tangent space at x is the dual

$$T^Z_{X,x} := \operatorname{Hom}(\Omega^Z_{X,x}, \mathbb{C})$$

Proposition 0.3. *The Zariski tangent space and the ordinary tangent space are isomorphic.*

Proof. Let D be a derivation. We define a map $\psi_D : \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}$ by $\psi_D(f) = D(f)$. As before, this is well defined, because $fg \mapsto f(x)D(g) + g(x)D(f)$, and our $f, g \in \mathfrak{m}$, so $f(x) = g(x) = 0$. Conversely, we define a map $\mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{C}) \to T_{X,x}$ by

$$\phi \mapsto D_\phi$$

where $D_\phi(f) = \phi(f - f(x))$[1]. This is a derivation by a standard trick: we have

$$\begin{aligned} D_\phi(fg) &= \phi(fg - f(x)g(x)) \\ &= \phi(fg - fg(x) + fg(x) - f(x)g(x)) \\ &= \phi(f(g - g(x)) + g(x)\phi(f - f(x)) \\ &= \phi(f(g - g(x)) + g(x)D_\phi(f) \end{aligned}$$

and we must deal with $\phi(f(g - g(x))$. But once again, we have

$$f(g - g(x)) = (f - f(x))(g - g(x)) + f(x)(g - g(x))$$

and thus

$$\phi(f(g - g(x)) = \phi((f - f(x))(g - g(x))) + \phi(f(x)(g - g(x))) = 0 + f(x)\phi(g - g(x))$$

since $(f - f(x))(g - g(x)) \in \mathfrak{m}^2$. Thus, we have shown that

$$D_\phi(fg) = f(x)D_\phi(g) + g(x)D_\phi(f)$$

i.e. D_ϕ is a derivation, as claimed. It is easy to see that $D_{\psi_D} = D$ and $\psi_{D_\phi} = \phi$, i.e. the two maps are inverses. ♢

Corollary 0.4. *We have* $\Omega_{X,x} = \Omega^Z_{X,x} = \mathfrak{m}/\mathfrak{m}^2$.

Definition 4. Let $X = \operatorname{Spec} A \subset \mathbb{A}^n$ be a variety of dimension k. We say X is non-singular or smooth at x if $\Omega^Z_{X,x}$ is k-dimensional.

Definition 5. The Jacobian matrix $J(x)$ is the $m \times n$ matrix

$$(\partial f_i/\partial x_j(x))_{ij}$$

Theorem 0.5. *Let* $X = \operatorname{Spec} A = \operatorname{Spec} \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$, $x \in X$ *as above. Suppose* X *is* k*-dimensional. Then* X *is non-singular at* x *if and only if* $J(x)$ *has rank* $n - k$.

Proof. Let $\mathfrak{n}$ be the ideal of x in $\mathbb{A}^n$. Write $I = (f_1, \cdots, f_m)$ for the ideal of X. Taking the quotient map $k[x_1, \cdots, x_n] \to I$, we see that there is a map

$$\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$$

i.e. a map $\Omega^Z_{\mathbb{A}^n,x} \to \Omega^Z_{X,x}$. It is easy to see that $\Omega^Z_{\mathbb{A}^n,x}$ is n-dimensional, generated by the functions $x_i - a_i$ where $x = (a_1, \cdots, a_n)$. These correspond to the Kahler differentials dx_i. We have

$$\mathfrak{m}/\mathfrak{m}^2 = (\mathfrak{n}/I)/(\mathfrak{n}^2 + I/I) = \mathfrak{n}/(\mathfrak{n}^2 + I)$$

and so the kernel of the map $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is precisely I. Under our isomorphism above, an element $f_i \in I \subset \mathfrak{n}$ goes to the element

$$df_i$$

But now I claim that the simple expression you'd expect from calculus holds:

$$df_i = \sum \frac{\partial f_i}{\partial x_j}(x)dx_j$$

[1] note $f - f(x) \in \mathfrak{m}$

Simply, by Taylor expansion, we have

$$f_i(x_1, \cdots, x_n) = f_i(x) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)(x_j - a_j) + \text{terms of higher order in } (x_j - a_j)$$

But $f_i(x) = 0$ since the f_i by assumption vanish at x, and given a higher order term its differential will be divisible by some $(x_j - a_j)$, so vanishes at x. So the differential reduces to the expression claimed.

Thus, we see $\Omega^Z_{X,x}$ is k-dimensional if and only if the space generated by the df_i has dimension $n - k$. This is exactly the rank of the Jacobian. $\Diamond$