Toric Geometry

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November 28, 2023

Abstract

This paper are the notes for our seminar on the 28.11.2023. We will talk about differential forms on toric varieties.

1 Differential forms on smooth varieties

Let us start with some notation.

Definition 1.1 *Let X be a smooth variety of dimension n. Denote the locally free sheaf of rank k on X as $\Omega_X^{\wedge k}$. Differently said, the sheaf of k-forms. Further, we set $\Omega_X := \Omega_X^{\wedge 1}$.*

Remark 1.2 *Assume that there is an open cover for $X = \bigcup_i U_i$. Then, we have that*

$$\Omega_X|_{U_i} \simeq \mathcal{O}_{U_i}^n$$

with $\mathcal{O}_{U_i} = \{f : U_i \to \mathbb{C}\}$ So if $z_1, ..., z_n$ are coordinates around a point $x \in X$, we have the localization

$$\Omega_{X,x} = < dz_1, ..., dz_n >= \{\sum f_i dz_i | f_i \in \mathcal{O}_{X,x}\}$$

Before we can start with a first example, another notation must be introduced.

Definition 1.3 *We set $\mathcal{O}(D)$ as the functions with poles of order at worst of D. It is common to introduce a minus $\mathcal{O}(-D)$. In this case, we consider functions with zeros of minimal order D.*

Remark 1.4 *Note the difference between $\mathcal{O}(D)$ and $\mathcal{O}_{U_i}$.*

Example 1.5 *Let us consider $\mathbb{P}^1$. We have seen that in this case*

$$\Omega_X = \mathcal{O}(n)$$

This n can be computed via the following argument: Set

$$U_1 = \{\frac{x_1}{x_0} \quad | \quad [x_0 : x_1] \in \mathbb{P}^1, x_0 \neq 0\} \simeq \mathbb{C}_z$$

$$U_2 = \{\frac{x_0}{x_1} \quad | \quad [x_0 : x_1] \in \mathbb{P}^1, x_1 \neq 0\} \simeq \mathbb{C}_w$$

where $z = \frac{1}{w}$

Now, let us take the differential 1-form dz on U_1 and the differential 1-from $f(w)dw$ on U_2. Observe that

$$dz = \frac{-1}{w^2} dw$$

Now, if we want to get a well defined differential on $\mathbb{P}^1$, we have to check, whether the two differentials coincide on $U_1 \cap U_2$. This is only possible, if the function $f(w)$ is set to be $f(w) = \frac{-1}{w^2}$. This is a function with a pole of degree 2.

Interestingly, it can be shown that no matter how we choose our differential on U_1, this function will have a pole of order 2. Therefore, we have showed that $n = -2$.

Now, we aim to understand $\Omega_X^{\wedge n}$ a little better. Let us consider with a proposition.

Proposition 1.6 *Let X be smooth. Then there exists a divisor D :*

$$\Omega_X^{\wedge n} = \mathcal{O}(D)$$

Proof 1.7 *The basic idea is that $\Omega_X^{\wedge n}$ has rank 1 and must be a line bundle.*

2 Differential forms on toric varieties

2.1 The locally free sheaf of differentials

In case of a toric variety, there exists a proposition that tells us what this divisor must be. But first a short reminder:

Remark 2.1 *Let $X(\triangle)$ be a smooth toric variety. Denote the to $\rho_i \in \triangle(1)$ corresponding divisor as D_i. Then every divisor D on X can be written as*

$$D = \sum_{\rho_i \in \triangle(1)} a_i D_i$$

with $a_i \in \mathbb{Z}$

Now, we come to the first core proposition of today's seminar.

Proposition 2.2 *Let $X(\triangle)$ be a smooth toric variety. Then*

$$\Omega_X^{\wedge n} = \mathcal{O}(-\sum_{\rho_i \in \triangle(1)} D_i)$$

Example 2.3 *Consider $\mathbb{P}^1$ again. Then $\triangle(1) = \{1, -1\}$ We directly can compute:*

$$\Omega_X^{\wedge n} = \Omega_X = \mathcal{O}(-D_1 - D_2) = \mathcal{O}(-[0] - [\infty]) \simeq \mathcal{O}(-2)$$

Example 2.4 *Consider $\mathbb{P}^2$. Then $\triangle(1) = \{(1,0),(0,1),(-1,-1)\}$. Then*

$$\Omega_X^{\wedge n} = \mathcal{O}(-D_1 - D_2 - D_3) \simeq \mathcal{O}(-3)$$

Let us continue we the proof of the stated proposition.

Proof 2.5 *Let $e_1, ..., e_n$ be the generators of the lattice N. Denote $X_i = \chi^{e_i^*}$ and define the rational section*

$$\omega = \frac{dX_1}{X_1} \wedge ... \wedge \frac{dX_n}{X_n}$$

Observe, that another choice of basis for N gives the same differential form, up to multiplication by ± 1. We must show that $div(\omega) = -\sum D_i$. On an open set U_σ, we may assume σ is generated by part of a basis, say $e_1, ..., e_k$, so we have

$$U_\sigma = Spec(\mathbb{C}[X_1, ..., X_k, X_{k+1}, X_{k+1}^{-1}, ..., X_n, X_n^{-1}])$$

and

$$\omega = \frac{\pm 1}{X_1 \cdots X_n} dX_1 \wedge ... \wedge dX_n$$

This shows that $div(\omega)$ and $-\sum D_i$ have the same restriction to U_σ.

2.2 The locally free sheaf of differentials with logarithmic poles

Besides Ω_X there exists the free sheaf $\Omega_X(logD)$ of differentials with logarithmic poles along D. Interestingly, these are trivial on a toric variety.

Remark 2.6 *At a point x in $D_1 \cap ... \cap D_k$ with x not in the other divisors, if $X_1, ..., X_n$ are local parameters such that $X_i = 0$ is a local equation for $D_i, 1 \leq i \leq k$, then*

$$\frac{dX_1}{X_1}, \frac{dX_2}{X_2}, ..., \frac{dX_k}{X_k}, dX_{k+1}, ..., dX_n$$

give a basis for $\Omega_{X,x}(logD)$

Let us continue with a short introduction of an interesting map.

Remark 2.7 *Consider the map*

$$\Omega_X(logD) \to \bigoplus_{\rho \in \triangle(1)} \mathcal{O}_{D_i}$$

$$\omega = \sum_i f_i \frac{dX_i}{X_i} \mapsto \bigoplus f|_{D_i}$$

Example 2.8 *Let $X = \mathbb{C}$. Then a divisor D is the sum of points. If you have a point p, then $\mathcal{O}_D = \mathcal{O}_p = f(p)$. So we would have the map*

$$f\frac{dz}{z} \mapsto f(p)$$

which seems familiar from the residue theorem in complex analysis.

This map comes up in the second core proposition of today's talk.

Proposition 2.9 *1. There exists an exact sequence of sheaves*

$$0 \to \Omega_X \to \Omega_X(logD) \to \bigoplus_{i=1}^{d} \mathcal{O}_{D_i} \to 0,$$

2. $\Omega_X(logD) \simeq M \otimes \mathcal{O}_X$

Proof 2.10 *To show the second point, consider the canonical map*

$$M \otimes_{\mathbb{Z}} \mathcal{O}_X \to \Omega_X(logD)$$

$$(u, f) \mapsto f \sum a_i \frac{dX_i}{X_i}$$

with $u = \sum a_i e_i^*$ By the remark from above, one can see that this is indeed an isomorphism.
For the first point, consider the residue map. It is zero precisely when each f_i is divisible by X_i, i.e, when ω is a section of Ω_X.